Probability Level 1

Terminology

Probability measures the chance something happens, in numbers.

It measures how likely is the outcome (result) of an event. We write P(result) as a shorthand.

An event is some measurable test, measurement or observation. Sometimes trial is used instead.

Mutually exclusive results are the exact opposites of each other: either one or the other must happen. The use of “not” helps with mutually exclusive events: the opposite of “rainy” is best thought of as “not rainy” rather than vague terms such as “fine”.

Another formal term seen sometimes is “complement” for the opposite of an outcome. The complement is marked with a dash, ' , and is the chance of the outcome **not** happening.

So we can write that $P(A) + P(A') = 1$

Numerical value

Probability is a number between 0 and 1.

$P(X) = 0$ means that the outcome X is impossible

$P(X) = 1$ means that the outcome X is certain.

Probabilities may be given in any appropriate form: decimal, fraction or percentage.

A probability of 0.2 means very little, but $\frac{1}{5}$ makes it much more obvious that it is one chance in five.

Percentages are awkward, because they have to be reconverted back to decimals before use – e.g. 55% of 120 is 0.55×120.

Students need to leave descriptions such as “likely” behind. They are sometimes useful to describe what a probability value means, but they are not suitable when asked for a probability.

No probability can be bigger than one. If students get an answer larger than 1 then they should immediately recognise that as impossible.

Experimental Probability

Probabilities can be calculated experimentally. A number of trials are conducted (or events are measured) and

$$Probability = \frac{number\ of\ successes}{number\ of\ trials}$$

Experimental probability is only an estimate, and will change slightly each time it is measured.

When calculating experimental probability we ideally test the entire population, which is called a "census" to get an exact result.

Often a census is not possible because it is too difficult, too expensive etc or the population is infinite. In those cases we take a sample and work from that. The sample needs to be:

– random and unbiased

– independent (the outcome of one trial does not affect the result of the others)

– sufficiently large (how large depends on how accurate you need it to be)

Students may notice that these conditions are the same as those for proper sampling for all statistics.

The sample can be a series of experiments or trials.

Because experimental probability is always only an estimate, students should not expect their results to match with any theory calculation. They should **not** under any circumstance "fix" the outcome of their experiments to match what they think should happen.

If an experiment returns an unexpected result, run it again. A second run will normally be closer.

Any small sample can always be very different from the true probability, but over a long run of experiments, the experimental probability will get closer and closer to the true experimental probability.

The ability to express the concept of experimental variation clearly is worth an important skill for Merit and Excellence.

- each sample can give different results, due to randomness;
- even well run experiments or well-taken samples can occasionally give wildly different results from theory;
- only an experiment with a very large number of trials or a very large sample will give a reliable experimental probability, but even then it will normally only be close to the correct answer, not exact.

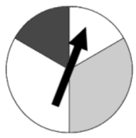
Calculating Simple Theoretical Probability

Theoretical probability can be calculated using a similar calculation to experimental probability.

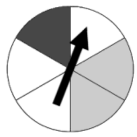
So long as each result (outcome) is equally likely:

$$Probability = \frac{successful\ results}{all\ possible\ results}$$

When calculating theoretical probability it is important to divide up the outcomes in such a way as to make them equally likely.



needs to be thought of as



so that the white is $\frac{3}{6}$ of the area.

The probability of white is $\frac{3}{6}$.

The combination of two outcomes combined can usually be calculated by a simple table, provided each of the columns is equally likely and each of the rows is equally likely.

The sum of rolling a four-sided dice and a six-sided dice can be made into a table:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10

The probability of rolling a sum of 8 with a 4-sided and a 6-sided dice is therefore three chances in 24, or 12.5%, because the table shows 24 equally likely outcomes, and three of them are totals of eight.

If the various outcomes cannot be made equally likely, or if the result of one trial affects the probability of the next, then more complicated procedures must be used. At Year 11 this means using probability trees (although in later years more techniques are introduced).

Expected Value

The expected value is the number of successes that are expected for a certain number of trials.

Expected value = probability × number of trials

The expected value is basically the same as the mean (average) number of successes, except that a mean can be a decimal value, whereas an expected value is rounded because it is not possible to have a fractional success.

The probability of 6 on a normal dice is = $\frac{1}{6}$. If we roll the dice 20 times we calculate

$\frac{1}{6} \times 20 = 3.33$ sixes on average,

but the expected number of sixes is 3, since you can't get $\frac{1}{3}$ of a roll.

Calculations with Probabilities

If two results are possible from a single trial, then the probability of either is the probability of each taken alone added together.

e.g. if a test has three possible outcomes: A, B or C, then for any one trial we can say that

P(A or B) = P(A) + P(B)

If two results occur in separate trials, then the probability of one then the other is the probability of each taken alone multiplied together.

e.g. if a test has three possible outcomes: A, B or C, then we can say that

P(A both times) = P(A) × P(A)

It is important when calculating probabilities to consider different arrangements which give the same result as effectively different.

e.g. if we toss two coins, then there are three possible outcomes (two heads, two tails, one of each) but we have to consider HT and TH as different. It can help to consider the coins as different colours, so that red H and blue T is obviously different to red T with blue H.

The probability of one head and one tail from a toss of two coins is $\frac{2}{4}$, not $\frac{1}{3}$.

Probability calculations are made much easier if real life differences which do not matter to the question are ignored.

e.g. when calculating the probability of twice drawing a card from a standard deck without getting a heart it is not helpful to count clubs, diamonds or spades as separate options – for this purpose what matters is that the card is either a heart or not a heart.

P(no hearts) = ¾ × ¾ = 0.75 × 0.75 = 0.5625

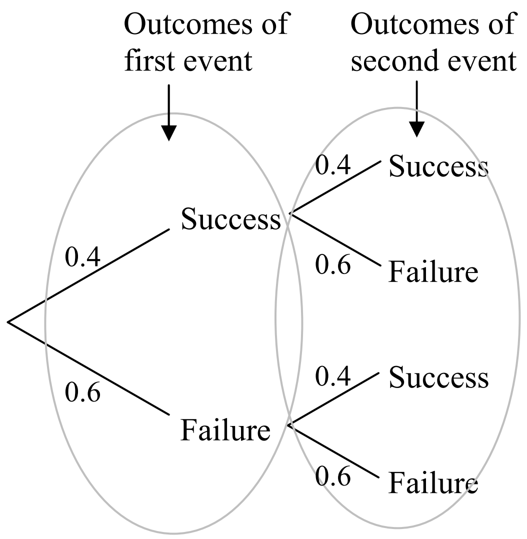
Probability Trees

Trees are generally the best way to resolve complicated problems with multiple paths to success, when the various outcomes are not equally likely, or when the outcome of the first event changes the probability of the second event.

- Start from what is actually happening. You need to be very clear on this.
- What are the potential results of the event you consider first? Draw branches to those outcomes.
- Carry on for each following event from the ends of the previous branch. What potential results can follow? Draw branches to those outcomes.
- When finished with the tree, write on each branch its probability.
- Decide which paths down the branches give the results you are looking for.
- The probability of each result is the probabilities on that branching multiplied.
- The final probability is the sum of all the individual paths of the result of interest.

Note: an <u>event</u> is some measurable test or observation. The results of an event are its <u>outcomes</u>.

Each vertical row of a tree represents a separate event: something that happens with a measurable outcome (result). The branches go to those different outcomes, each with its probability attached.



Probability

P(S, S) = 0.4 × 0.4 = 0.16

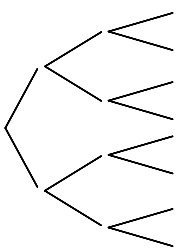
P(S, F) = 0.4 × 0.6 = 0.24

P(F, S) = 0.6 × 0.4 = 0.24

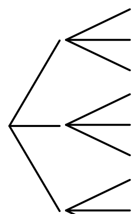
P(F, F) = 0.6 × 0.6 = 0.36

Thus the probability of getting one, and only one, success is = 0.24 + 0.24 = 0.48

The key is getting the structure right first, before worrying about the rest. Each event adds another family of branching, while more outcomes at each event means more lines at each branching.



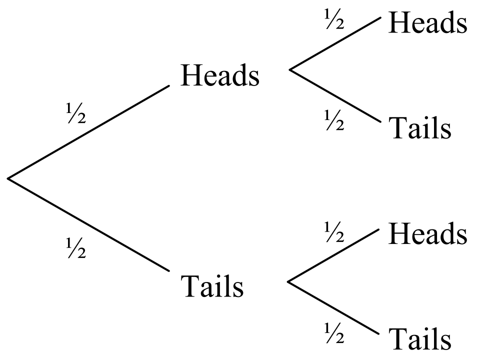
Three events, each with two outcomes



Two events, each with three outcomes

Generally when drawing trees it is wise to work through events in the order they occur, as it is less confusing. If events occur simultaneously then it does not matter which order they are drawn, but separate events must always have separate branching, even when they occur at the same time.

e.g. if two coins are tossed at the same time, the tree is



P(2H) = 0.5 × 0.5 = 0.25

P(H, T) = 0.5 × 0.5 = 0.25

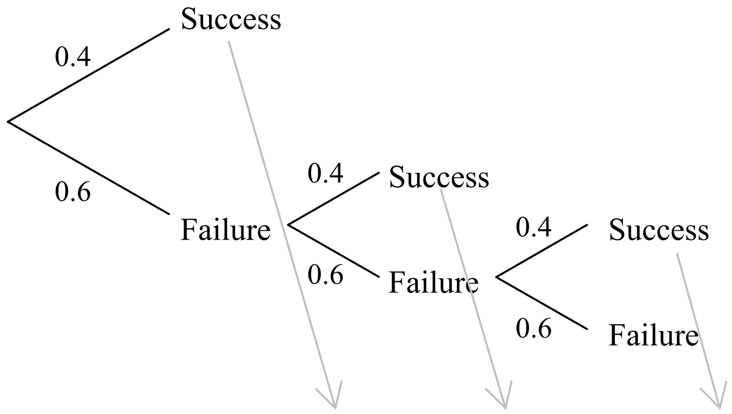
P(T, H) = 0.5 × 0.5 = 0.25

P(2T) = 0.5 × 0.5 = 0.25

Note that although the tosses are at the same time, it is important to distinguish the two different ways a head and a tail can be reached. It can help to imagine the coins are different colours, so that red H and blue T is obviously different to red T with blue H.

It is not necessary to draw every branch in many cases. Once further branching has no meaning, either because success or failure is automatic, it can be stopped.

e.g. if we want to draw a tree to see if three trials occur with at least one success, then once any success is reached there is no need to continue the tree further. The tree becomes:



P(at least one success) = (0.4) + (0.6 × 0.4) + (0.6 × 0.6 × 0.4) = 0.784

Often it is quicker to work out answers by looking at the complementary (opposite) result.

In the case above, P(at least one success in three trials) = 1 – P(all three are failures)

= 1 – (0.6 × 0.6 × 0.6) = 0.784

Sampling Without Replacement

Most questions at Year 11 do not need this, but it becomes increasingly important later.

> If a selection leaves the pool of options changed, then each following selection is at slightly different odds.
>
> The key is to calculate the probability of each step separately.
>
> $Probability = \frac{successful\ results}{all\ possible\ results}$ as before, but now the number of results changes with each event.

After each step the number of possible results drops by one. The successful result also drops by one if a success occurred previously, because that option is no longer possible.

There are 10 cards, of which five are red. Two are drawn at the same time.

P(1^{st} is red) = the Probability the first card drawn is red = 5 out of 10.

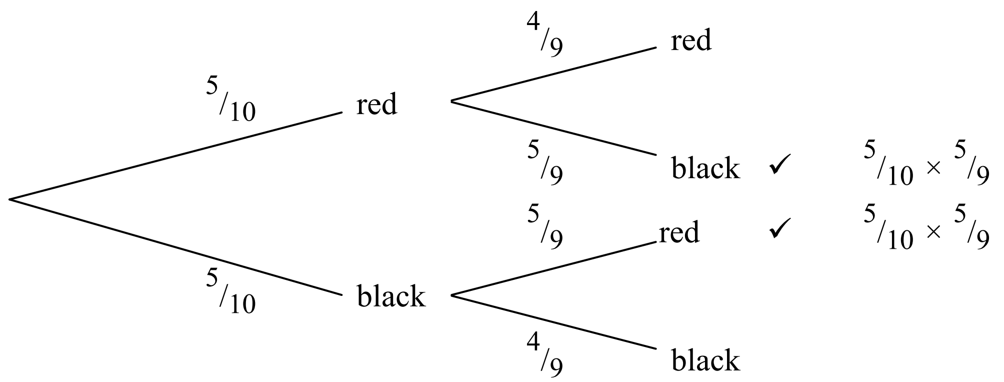
P(2^{nd} is red) = 4 out of 9, since one of the five red cards is gone, out of the 9 cards left

P(2 reds) = P(1^{st} is red) × P(2^{nd} is red) = $\frac{9}{10} \times \frac{4}{9}$ = 0.2

Things are slightly more difficult if the probability required mixes some successes and some failures. Each step must be thought through on its own.

There are 10 cards, of which five are red. Two are drawn at the same time.

Seeking the probability that they are different colours.



P(different colours) = $^5/_{10} \times {}^5/_9 + {}^5/_{10} \times {}^5/_9 = {}^5/_9 =$ 55.56%