Level 1 Trigonometry

Trigonometry is mostly about following a set of rules, which need to be memorised properly before anything else is attempted.

The steps for any trigonometry problem are always the same:

1. Find a right angle triangle to solve.
2. Name the sides using the conventions, given below.
3. If no angle is involved, used Pythagoras' Theorem, otherwise $S\frac{O}{H}\quad C\frac{A}{H}\quad T\frac{O}{A}$
4. Choose the formula that gives uses the information that you have been given.
5. Put the values into the formula and solve.
6. Make sure the answer looks right.

If the triangle does not have a 90°, then you cannot use the rules of trigonometry given here. Instead the problem must be converted into one where there is one, or more, right angle triangles. This often involves some geometry.

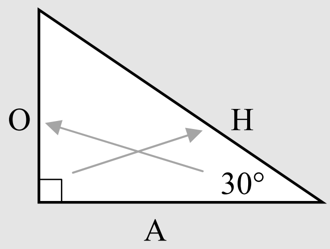
Naming Convention

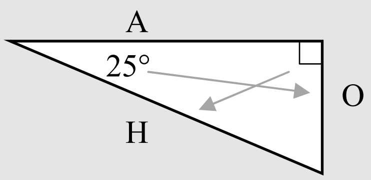
Naming the sides correctly is key to Trigonometry, as picking the correct formula depends on it.

The long side of any right angle triangle is called the Hypotenuse.

The hypotenuse is always the side directly across from the right angle.

If another angle is given or wanted, then the side opposite that is called the Opposite, and the side touching it is called the Adjacent.





The Hypotenuse can be the top, bottom, left or right side: what matters is that it is always opposite the right angle.

Always label the Hypotenuse first. This prevents confusion about which is the Adjacent, and makes sure that questions are not attempted on triangles without a right angle.

Pythagoras Theorem

For any right angled triangle: the long side squared = short side squared + other short side squared

$H^2 = a^2 + b^2$

When finding a short side the formula rearranges to give:

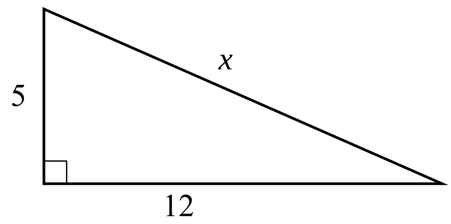
$a^2 = H^2 - b^2$ or $b^2 = H^2 - a^2$

It does not matter which short side is designated "a" and which is "b" for Pythagoras.

It is extremely helpful to set up a layout and routine which you use every time you do a Pythagoras question, as that helps avoid mistakes. The process cannot be short cut. Each side must be 1) squared, 2) added or subtracted depending on whether you want long or short side, then 3) square rooted in order to get the answer.

Finding a long side requires a bigger number than the other sides, so you need to add the squares. Finding a short side means that the number must be smaller than the long side, so you need to subtract the squares.

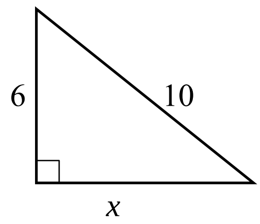
Until you are very comfortable, actually writing out the formula at the start of every exercise is worthwhile. It helps you learn it better, and reminds you whether you are adding or subtracting.



Long side: $H^2 = a^2 + b^2$

$x^2 = 12^2 + 5^2 = 169$

$x = \sqrt{169} = 13$



Short side: $a^2 = H^2 - b^2$

$x^2 = 10^2 - 6^2 = 64$

$x = \sqrt{64} = 8$

A common mistake is to forget to take the square root after adding or subtracting the squares. This can be avoided by (1) always doing your working in the same routine, so it becomes automatic, and (2) always checking your answer to see if it makes sense. A ridiculously large answer suggests that you have forgotten to take the square root.

If you get a negative answer and try to square root it, you will get a Maths error. You always take the short side squared from the Hypotenuse squared ($a^2 = H^2 - b^2$ or $b^2 = H^2 - a^2$) to avoid this.

Pythagoras Theorem is often seen as $a^2 + b^2 = c^2$, but that tends to hide which side is the long side and make it easier to get wrong. It is also confusing in the 3-dimensional case. Use $a^2 + b^2 = H^2$.

Finding a Side Length with Sin/Cos/Tan

After sorting out the 90° triangle you wish to solve and labelling it, you will be left with a side you need to find and a side length you are given. You select your formula from SOHCAHTOA based on which letter triangle has the side you are given and the side you want.

Write them as triangles: $\overset{O}{S\quad H}$ $\quad$ $\overset{A}{C\quad H}$ $\quad$ $\overset{O}{T\quad A}$
where: H = Hypotenuse, A = Adjacent, O = Opposite, S = Sine, C = Cosine, T = Tangent

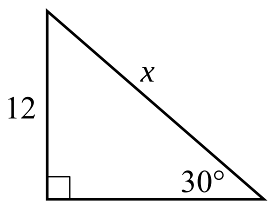
With the selected formula, cover the letter being found and that gives the relationship needed.
It is a division if the uncovered letters are on different levels:

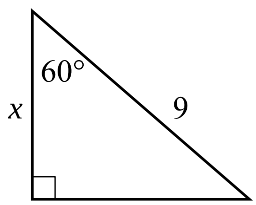
$H = \frac{O}{S}$ $\qquad$ $H = \frac{A}{C}$ $\qquad$ $A = \frac{O}{T}$

It is a multiplication if the uncovered letters are side by side:

$O = S \times H$ $\qquad$ $A = C \times H$ $\qquad$ $O = T \times A$

It is useful to write out the part of SOH CAH TOA being used every time, in order to prevent confusion about what number goes where. Setting up a regular routine and layout is a useful way of ensuring that you make less mistakes and can remember the process later.





We need the H (x) and have the O (12)

That means we use $\overset{O}{S\quad H}$

$H = \frac{O}{S} = \frac{12}{\text{Sin } 30} = 24$

We need the A (x) and have the H (9)

That means we use $\overset{A}{C\quad H}$

$H = C \times H = \text{Cos } 60° \times 9 = 4.5$

In SOH CAH TOA the "S" is the Sine (SIN) function, the "C" is the Cosine (COS) function and the "T" is the Tangent (TAN) function. These operate only on the **angles**. It is vital that you do not attempt to take the sin/cos/tan of the side length.

Watch that most modern calculators require brackets to separate the angle portion from any calculation. Above, where we want Cos 60° × 9 we need to type in cos(60) × 9 into the calculator.

Up to Year 11 trigonometry is always done using degrees, but your calculator also operates in radians and gradians. You need to ensure it is set correctly – shown by a little "D" at the top of the screen – every time memory is cleared.

Finding an Angle with Sin/Cos/Tan

After sorting out the 90° triangle you wish to solve and labelling it, you will be left with two sides given and an angle to find.

As before select your formula from $\mathrm{S}\overset{\mathrm{O}}{\quad}\mathrm{H}$ $\quad\mathrm{C}\overset{\mathrm{A}}{\quad}\mathrm{H}$ $\quad\mathrm{T}\overset{\mathrm{O}}{\quad}\mathrm{A}$ based on the two sides.

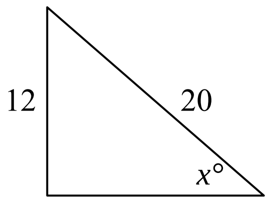
The relationship is given by a division:

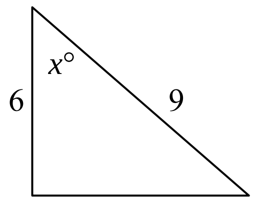
$\text{Sin(angle)} = \frac{\mathrm{O}}{\mathrm{H}}$ $\qquad\text{Cos(angle)} = \frac{\mathrm{A}}{\mathrm{H}}$ $\qquad\text{Tan(angle)} = \frac{\mathrm{O}}{\mathrm{A}}$

But we need the angle, not the trig(angle), so we move the function across the equals sign and do the opposite:

$\text{angle} = \sin^{-1}\left(\frac{\mathrm{O}}{\mathrm{H}}\right)$ $\qquad\text{angle} = \cos^{-1}\left(\frac{\mathrm{A}}{\mathrm{H}}\right)$ $\qquad\text{angle} = \tan^{-1}\left(\frac{\mathrm{O}}{\mathrm{A}}\right)$

It is useful to write out the part of SOH CAH TOA being used every time. Doing this as part of a set routine and layout reduces mistakes and helps make the process automatic. When finding an angle it pays to write out that you are going to use the **inverse** form of sin, cos or tan.





We have the H (20) and O (12)

That means we use $\mathrm{S}\overset{\mathrm{O}}{\quad}\mathrm{H}$

$\sin(x) = \frac{\mathrm{O}}{\mathrm{H}}$

$x = \sin^{-1}\left(\frac{12}{20}\right) = 36.87°$

We have the H (9) and A (6)

That means we use $\mathrm{C}\overset{\mathrm{A}}{\quad}\mathrm{H}$

$\cos(x) = \frac{\mathrm{A}}{\mathrm{H}}$

$x = \cos^{-1}(6 \div 9) = 48.19°$

The inverse functions, i.e. $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, are found on calculators by using the shift key before the normal sin, cos or tan key. You **cannot** type "cos–1" directly into the calculator. If the division is done as one step with the $\cos^{-1}$ etc, it needs to be bracketed on the calculator – so we need to type in shift-sin(12÷20) so that the entire division is acted on by $\sin^{-1}$.

It is vital to get your fraction the right way up and always writing out the formula helps ensure this.

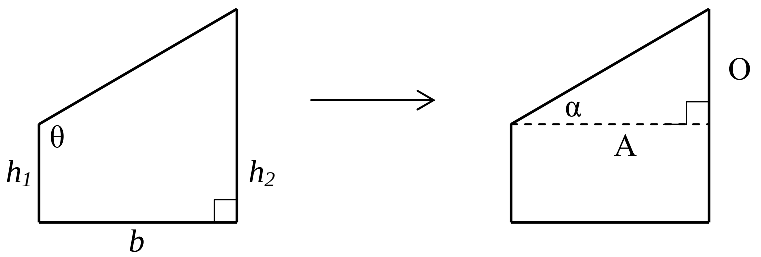
Often Greek letters are used to represent the unknown angles. Generally this is θ (theta), but also α (alpha), β (beta), γ (gamma) and the rest. It's just a letter representing an unknown value and you shouldn't get fixated on it being Greek letter. Make it *x* if it bothers you too much.

Using Geometry when the Triangle is not 90°

There are a fairly limited number of ways that Merit and Excellence questions can be posed, and students hoping for a good mark should be able to memorise all the possible options, more or less.

> The trigonometric relationships given by $S\frac{O}{H}$ $C\frac{A}{H}$ $T\frac{O}{A}$ only work for right angle triangles.
>
> To find the triangle often involves a little geometry.

A trapezium can be converted to rectangle and triangle.

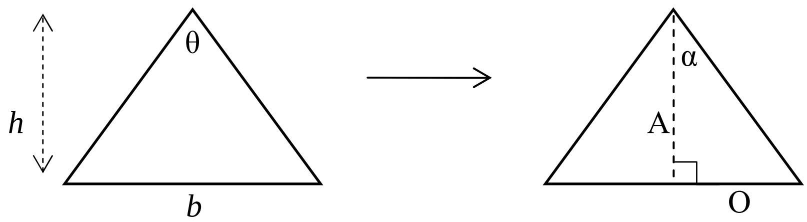


$\alpha = \theta - 90°$

$O = h_2 - h_1$

$A = b$

An isosceles triangle can be converted into two identical right angle triangles.

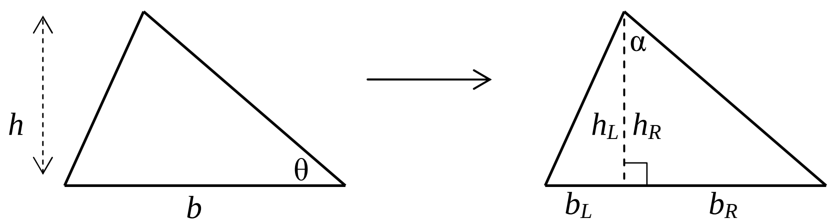


$\alpha = \frac{1}{2}\theta$

$O = \frac{1}{2}b$

$A = h$

Other triangles can be converted into two non-identical right angle triangles, who share a side.

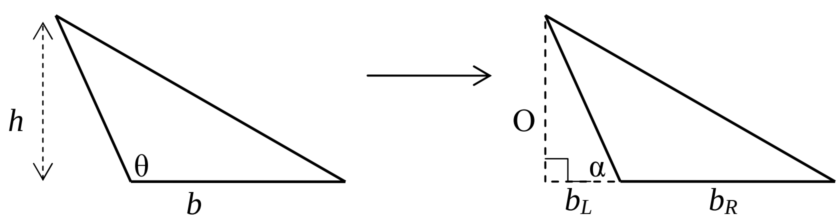


$\alpha = 90° - \theta$

$b_L + b_R = b$

$h_L = h_R = h$

This includes the situation where the original is obtuse, and the second triangle is **outside** the first.



$\alpha = 180° - \theta$

$b_R = b$

$O = h$

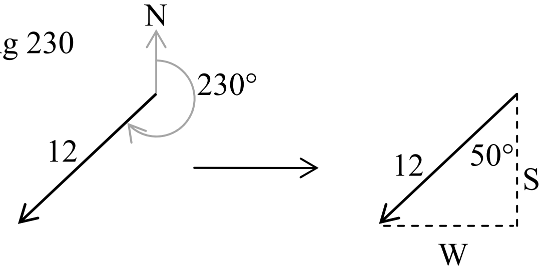
Using Geometry for Directions and Bearings

Merit and Excellence questions often involve directions.

A bearing is the angle clockwise from North. They are written in three digits, with no degree sign.

When a bearing is greater than 090 the triangle needs to be generated from S, W or E so that it is right angled.

If I go 12 km at a bearing 230



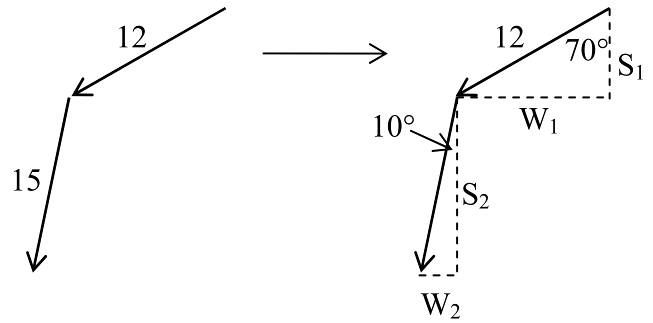
$S = \cos 50° \times 12$

$W = \sin 50° \times 12$

Any direction can be broken up into components as a North-South and East-West pair.

When adding two paths, we need to do this via their NS and EW components, which we put together at the end.

If I go 12 km at bearing 250, then 15 km at bearing 190.



The distance West of the starting point is: $W_1 + W_2 = \sin 70° \times 12 + \sin 10° \times 15$

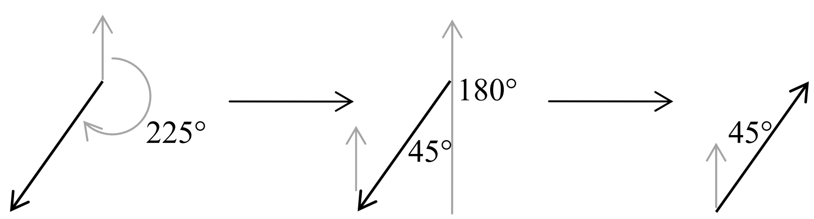
The distance South of the starting point is: $S_1 + S_2 = \cos 70° \times 12 + \cos 10° \times 15$

Overall distance is found using Pythagoras: $\text{Distance}^2 = (W_1 + W_2)^2 + (S_1 + S_2)^2$

Overall bearing taken is found using Trigonometry: $\theta = \tan^{-1}\left(\frac{W_1 + W_2}{S_1 + S_2}\right) + 180°$

Many bearing questions require the return bearing be calculated. These can be answered by parallel line geometry, on the basis that all Norths are parallel.

If I go out at bearing 225, then to return I must go at 045.



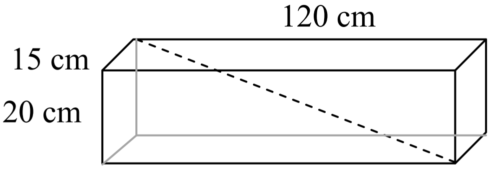
Questions in Three Dimensions

Pythagoras Theorem in 3D is $H^2 = a^2 + b^2 + c^2$, where a, b and c must be all at 90° to each other.

When dealing with the angle of a line to a plane, dropping directly down from the point is required.

The key to these is understanding the geometry, especially since any picture can only show the situation in two dimensions. Students need to take care to understand the situation being described fully before attempting to answer it.

The longest diagonal distance across a box is the standard 3D Pythagoras question. Each side is already at 90° to the others ("orthogonal") so the formula can be used directly.

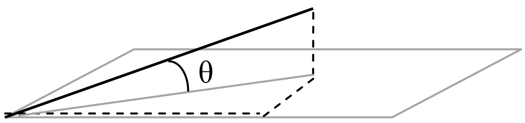


The dotted distance is

$= \sqrt{120^2 + 20^2 + 15^2} = 122.6$ cm

If the angle between a line and a plane is to be found, the "shadow" of the line on the plane (as if from a directly overhead light) is found first, and the angle calculated from that.

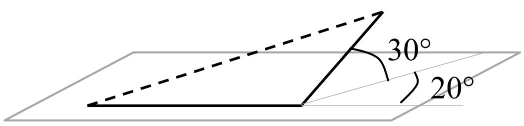
A line goes 4 in the *x* dimension, 3 in the *y* dimension, and 2 in the *z* dimension. Find θ.



The length of the "shadow" on the *x*–*y* plane is = $\sqrt{4^2 + 3^2} = 5$. So the angle formed with the plane, $\theta = \tan^{-1}\left(\frac{O}{A}\right) = \tan^{-1}\left(\frac{2}{5}\right) = 21.8°$

In the reverse situation, where the angle is given but only some of the dimensions, the process is again to work via the "shadow" on the plane given, generally breaking it into components.

A line goes along a plane 10 cm, then rises out of the plane 8 cm, at 30° to the plane and 20° to the first line. Find the dotted distance shown.

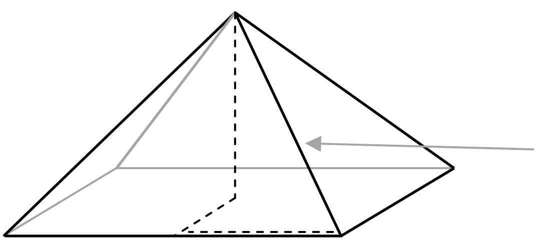


The second line has a vertical height from the plane = sin30° × 8 = 4 cm and a shadow at 90° to that **and** the first line = cos30° × 8 = 6.93 cm.

Dotted line = $\sqrt{10^2 + 4^2 + 6.93^2} = 12.8$cm

When working with pyramids the key geometric point is that each face is isosceles, so that halving it gets you to the midpoint (either across or deep, or both).

A pyramid has a base of 8 cm, and a height of 6 cm. Find the length of each rising edge.



The centre of the base is 4 cm along and 4 cm in from the edge. Using 3D Pythagoras, since we now have all the lengths at 90° to each other:

Length = $\sqrt{4^2 + 4^2 + 6^2}$
= 8.25 cm