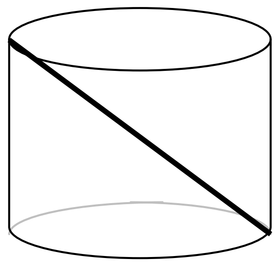
Y12 Calculus Excellence #2

1. The water in a tank is filling up at a rate of $\frac{dV}{dt} = 60 + 4t$ cubic centimetres per second.

   The tank is 25 000 cm^3 full after 10 seconds.

   Calculate how long it will be before the 100 000 cm^3 tank is full.

2. The diagonal length L of a cylinder is the length of the longest straight rod that will fit inside it. See the sketch

   Among all circular cylinders with diagonal length equal to 6, find the height of the one whose volume is largest.

   The volume of a cylinder = $\pi r^2 h$

3. It is believed that the velocity, V metres per microsecond, of an atomic particle t microseconds after collision is given by $V = kt^3$, where $k = 2.3\times10^{-4}$.

   How many metres will the particle travel in the fifth microsecond after collision?

4. On an island the number of crickets, C, depends upon the temperature, T.

   A scientist finds a curve he thinks will model the situation, which is:

   $$C(T) = 0.003(K - 6240T + 330T^2 - 5T^3)$$

   where K is a constant, T is in °C, and C is thousands of crickets

   He adjusts K so that the maximum number of crickets is 50 000.

   i) Find the minimum number of crickets in his model and the temperature that produces this minimum.

   ii) In what circumstances is his model likely to fail?

5. $g(x)$ is a parabola with a turning point at (2, 5) and a y-intercept of 7. What is its equation? (You must use *calculus* methods to answer this.)

6. A parabola that turns at $(3, k)$ has a gradient function that is a constant of 4. Write the equation of the parabola in terms of k.

7. A first drag car accelerates at a constant 5 ms^{-2}, and it starts 0.2 seconds after the starting green light. It is racing a second car with a constant acceleration of 6 ms^{-2}, but which starts 0.6 seconds from the green light. Where does the second car catch the first?

8. When is $y = 4x - 3x^2 - 2x^3 + 1$ a decreasing function?

Answers: Y12 Calculus Excellence #2

1. $V = 60t + 2t^2 + C$ by anti-differentiating

   We know $V = 25\,000$ at $t = 10$ $\Rightarrow$ $25000 = 60 \times 10 + 2 \times 10^2 + C$ $\Rightarrow$ $C = 24\,200$

   We need to solve: $100\,000 = 60t + 2t^2 + 24\,200$, quadratic – and ignore negative solution

   It takes 180 seconds

2. $V = \pi r^2 h$ and we need $V' = 0$, but we can't differentiate V with two variables (r and h)

   $\Rightarrow$ we need to substitute one out.

   Using Pythagoras $L = \sqrt{(d^2 + h^2)}$ where d = diameter (= 2r) and h = height

   Given $L = 6$ $\Rightarrow$ $6 = \sqrt{((2r)^2 + h^2)}$ $\Rightarrow$ $36 = 4r^2 + h^2$ $\Rightarrow$ $r^2 = 9 - 0.25h^2$

   $V = \pi r^2 h = \pi(9 - 0.25h^2)h = 9\pi h - 0.25\pi h^3$ so now we only have one variable for V

   $V' = 9\pi - 0.75\pi h^2$ $\Rightarrow$ $V' = 0$ when $9\pi - 0.75\pi h^2 = 0$ $\Rightarrow$ $9\pi = 0.75\pi h^2$

   $h = \sqrt{12}$ at maximum volume

3. $V = kt^3$ $\Rightarrow$ $S = 0.25kt^4 + C$ Distance (displacement) is anti-differentiating Velocity

   $S(4) = 0.25 \times 2.3 \times 10^{-4} \times 4^4 + C = 0.01472 + C$

   $S(5) = 0.25 \times 2.3 \times 10^{-4} \times 5^4 + C = 0.03594 + C$

   Distance in 5th microsecond is the difference from t = 4 to t = 5 (1st ms is t = 0 to t = 1)

   It will travel 0.02122 metres

4. $C(T) = 0.003(K - 6240T + 330T^2 - 5T^3)$ $\Rightarrow$ $C'(T) = {}^{-}18.72 + 1.98T - 0.045T^2$

   Turns when $C'(T) = 0$ $\Rightarrow$ $0 = {}^{-}18.72 + 1.98T - 0.045T^2$ $\Rightarrow$ $T = 30.25$ and 13.75

   As this is negative cubic, the maximum will be the right hand turning point = 30.25

   Put this into our equation to calculate K, noting that C is in thousands, so the max is 50

   $\Rightarrow$ $50 = 0.003(K - 6240 \times 30.25 + 330 \times 30.25^2 - 5 \times 30.25^3)$ $\Rightarrow$ $K = 41{,}859$

   The minimum is left turning point, and will give C(13.75)

   $C(13.75) = 0.003(41{,}859 - 6240 \times 13.75 + 330 \times 13.75^2 - 5 \times 13.75^3) = 16.355$

   i) **The minimum number of crickets is 16,355 at 13.75°C**

   The model peaks at 30° or so, which seems reasonable.

   The graph goes negative after 40°, which would just mean all the crickets die at that temperature, which seems OK.

   However the minimum is at 13.75, which leads to it getting bigger after that even as it gets colder. The y-intercept is $41{,}859 \times 0.003 = 125.5$

   ii) **The model fails for low temperatures, generating 125,000 crickets at 0°C.**

5. $g(x)$ is of form $g(x) = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$, as it is a parabola.

The y-intercept is 7, so c = 7.

Turns at (2, 5), so $g'(x) = 2\mathrm{a}x + \mathrm{b} = 0$ there $\Rightarrow g'(2) = 2\mathrm{a} \times 2 + \mathrm{b} = 0$ $\Rightarrow$ $\mathrm{b} = {}^{-}4\mathrm{a}$

So $g(x) = \mathrm{a}x^2 + ({}^{-}4\mathrm{a})x + 7$

Point $g(2) = 5 \Rightarrow$ $\mathrm{a} \times 2^2 + {}^{-}4\mathrm{a} \times 2 + 7 = 4\mathrm{a} - 8\mathrm{a} + 7 = 5$ $\Rightarrow$ $\mathrm{a} = 0.5$

$\boldsymbol{g(x) = \tfrac{1}{2}x^2 - 2x + 7}$

6. $f'(x) = 4x + \mathrm{a}$, as the gradient function's gradient is 4

$f'(3) = 0$, as it turns at $(3, k) \Rightarrow$ $f'(3) = 4 \times 3 + \mathrm{a} = 0$ $\Rightarrow$ $\mathrm{a} = {}^{-}12$

$f'(x) = 4x - 12$ $\Rightarrow$ $f(x) = 2x^2 - 12x + \mathrm{c}$

$f(3) = k$, as goes through $(3, k)$ $\Rightarrow$ $f(3) = 2 \times 3^2 - 12 \times 3 + \mathrm{c} = k$ $\Rightarrow$ $\mathrm{c} = k + 18$

$f(x) = 2x^2 - 12x + 18 + k$

7. $a_1 = 5$, so $v_1 = 5t + c$ Since the car starts from stopped c = 0

$s_1 = 2.5t^2 + c'$ Since the car starts from the zero point c' = 0

But we want our time zero to be the green light, so need to correct t

$s_1 = 2.5(t - 0.2)^2$ Since this makes t = 0.2 effectively zero for the car

$s_2 = 3(t - 0.6)^2$ By the same process but correcting for the 0.6 sec delay

They cross when $s_1 = s_2$ $\Rightarrow$ $2.5(t - 0.2)^2 = 2.5(t - 0.6)^2$

$\Rightarrow$ $2.5\,(t^2 - 0.4t + 0.04) = 3\,(t^2 - 1.2t + 0.36)$ $\Rightarrow$ $0.5t^2 - 2.6t + 0.98$

$\Rightarrow t = 0.4$ or 4.8 but the 0.4 solution is bogus because it ignores 0.6 starting time

Cars cross at $s_2 = 3(4.8 - 0.6)^2 =$ **52.9 metres from start**

8. $y = 4x - 3x^2 - 2x^3 + 1$ $\Rightarrow$ $y' = 4 - 6x - 6x^2$

The function is decreasing (going down) when the gradient is negative

So $4 - 6x - 6x^2 < 0$ $\Rightarrow$ ${}^{-}6(x - 0.4574)(x + 1.4574) < 0$

$(\div\ {}^{-}6)$ $\Rightarrow$ $(x - 0.4574)(x + 1.4574) > 0$ which is true when both brackets + or both –

Solution when $x > 0.4574$ and $x < {}^{-}1.4574$

If you struggle with solving the quadratic inequation, then calculate the turning points of y' and use your knowledge of graphs to say that a negative cubic is decreasing at its ends.