Non-Linear Graphs at Year 12

Students need to have a basic understanding of several key relationships, and the easiest way to do this is via their graphs. For students intending to take Level 3 Calculus this is even more important, as much of it will be assumed in Year 13.

In this booklet we look at:

- where one variable is squared relative to other – parabolas;
- where one variable is cubed relative to the other – cubics;
- where one variable is the inverse of the other – hyperbolas;
- where one variable is exponential to the other – exponential and log graphs;
- the concept of absolute value; and
- some work on terminology, especially functions.

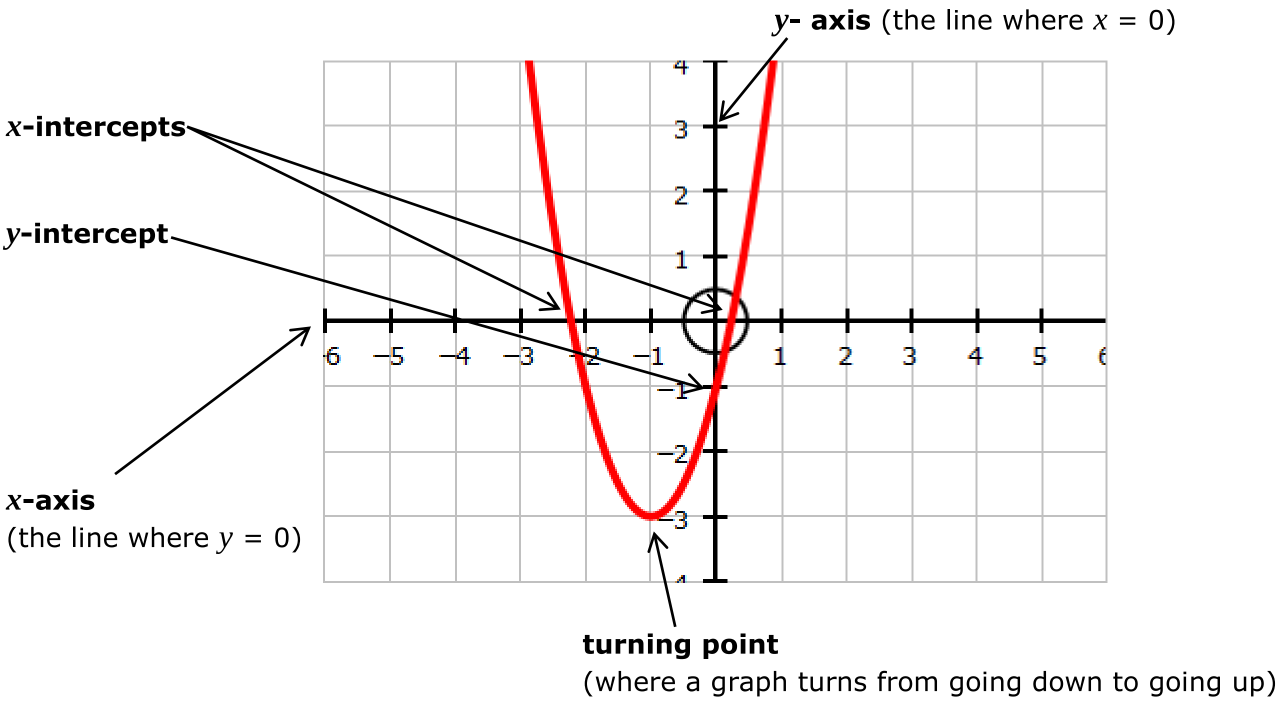
The checking of equations is far more effective if students have a graphics calculator. They need to know how to:

- write graphs into their calculator, correctly bracketed;
- use the graphical solve function to find key values; and
- change the view window to see the correct portion of the graph.

Again these skills will be assumed in Year 13 Calculus.

Key terminology

The following terms must be known:



Points on a graph are written in terms of (x, y) **co-ordinates**.

Parabolas

At Level 1 parabolas are written based on their x-intercepts and generally are very simple.

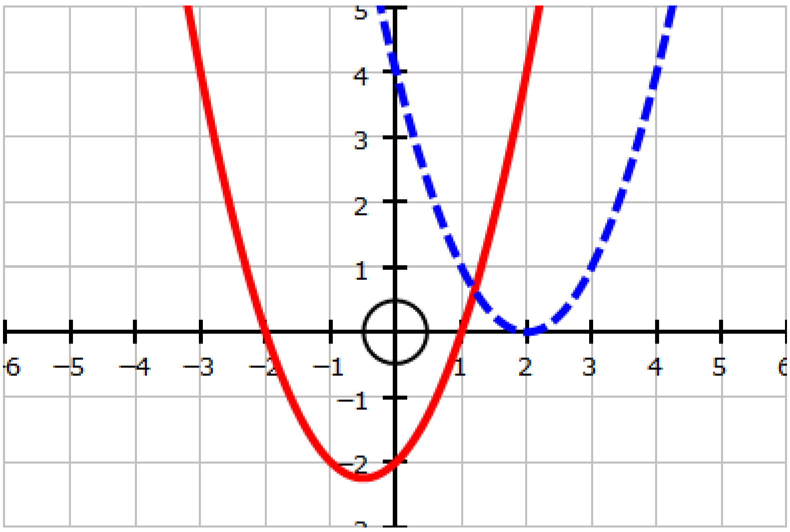
For example the two graphs here are

$$y = (x + 2)(x - 1)$$

and

$$y = (x - 2)^2$$

Recall that the graph passes through the x-axis at the solutions to y = 0, so the bracketed (factorised) versions use the negative of the solutions.



At Level 2 we must be able to deal with changing the steepness of the graphs.

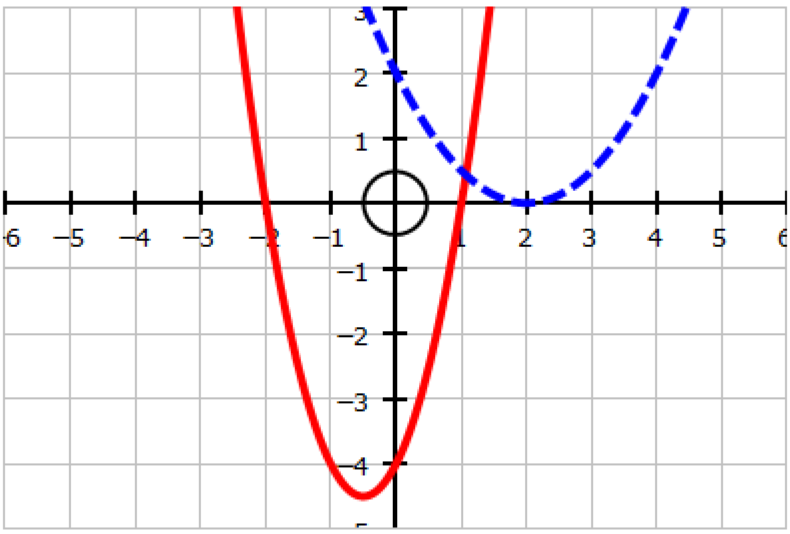
For example the two graphs here are

$$y = 2(x + 2)(x - 1)$$

and

$$y = 0.5\ (x - 2)^2$$

Note that the intercepts have not changed, just that the addition of the 2 × has made the solid one steeper and the 0.5 × has made the dotted one less steep.



At Level 2 we must also be able to write our equations based on where the turning point is.

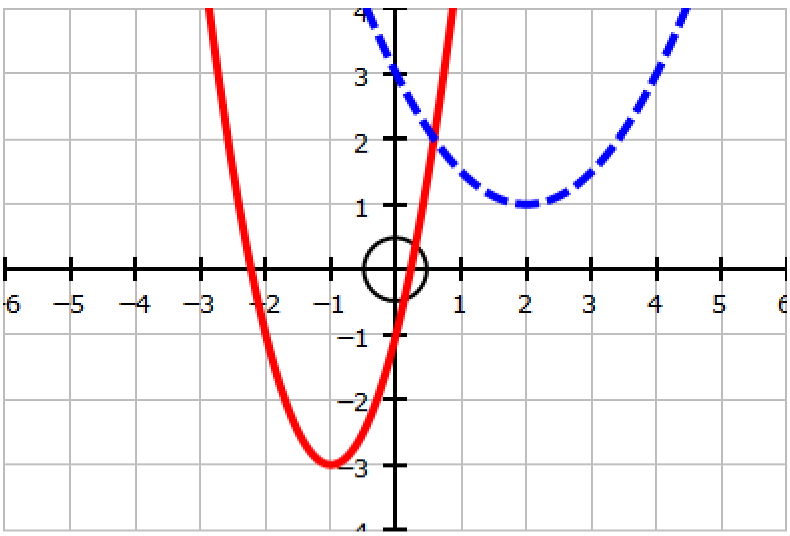
For example the two graphs here are

$$y = 2(x + 1)^2 - 3$$

and

$$y = 0.5\ (x - 2)^2 + 1$$

In both cases we are unable to use the intercept method, even if there are actual intercepts.



Turning Point Method

All parabolas can be written in the form: $\boldsymbol{y = m\,(x - t)^2 + c}$

where m determines the steepness of the graph;
t is the turning point's x value, but note we subtract it;
c is the vertical distance of the turning point from the x-axis.

1) Write out general form: $y = m\,(x - t)^2 + c$
2) Use the turning point to generate t and c. Remember to apply negatives to x shifts.
3) Use a known value to calculate m. The y-intercept is the best, if you have it.
4) Check the equation on your graphics calculator.

Examples

From the turning point at (2, −1) so

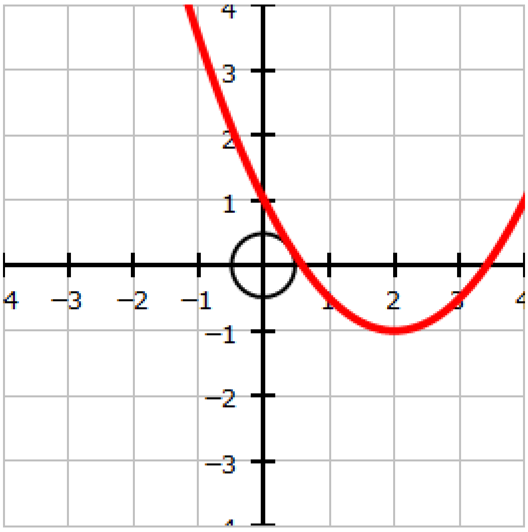
$$y = m\,(x - 2)^2 + -1$$

We see (0, 1) is on the graph, so:

$$1 = m\,(0 - 2)^2 - 1$$
$$2 = m\,(-2)^2$$
$$m = 0.5$$

The equation of this graph is $y = 0.5\,(x - 2)^2 - 1$

A quick check on the graphics calculator shows that this is correct.



It can help to think of the parabola starting as $y = x^2$, which turns at (0, 0), then "shift" the turning point in the x and y directions – adding a negative value to x and y to do this – before applying steepness multiplier, m.

Here the original $y = x^2$ turning point is moved up two and across negative one, which means we have a starting equation of:

$$y - (2) = m\,(x - (\text{-}1))^2$$

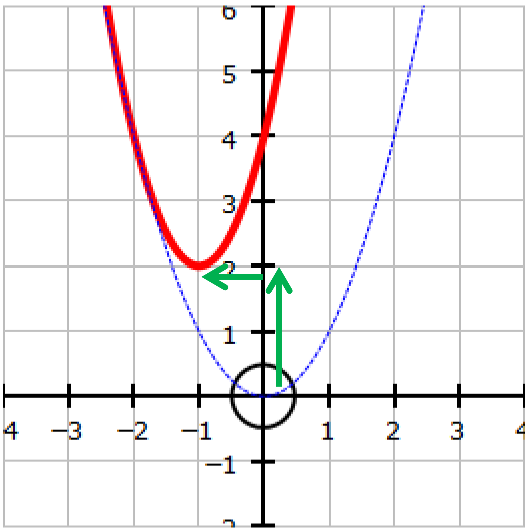
where x and y have the negative of the shifts, which on cancelling negatives and swapping the y shift across gives:

$$y = m\,(x + 1)^2 + 2$$

We know that (0, 4) is a point on the graph, so:

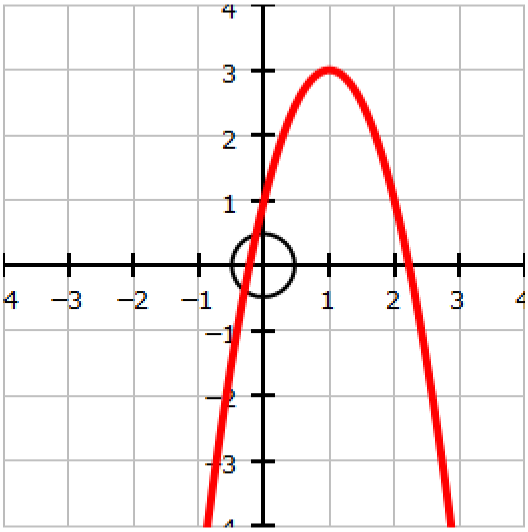
$$4 = m\,(0 + 1)^2 + 2$$
$$2 = m\,(1)^2$$
$$m = 2$$

And so we get $y = 2\,(x + 1)^2 + 2$



Variations

The m value does not have to be positive. If negative, the parabola is upside down.



The turning point here is at (1, 3) so

$$y = m\,(x - 1)^2 + 3$$

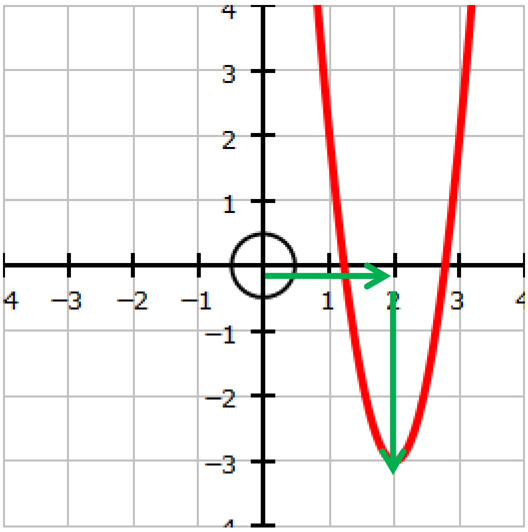
We see (0, 1) is on the graph, so:

$$1 = m\,(0 - 1)^2 + 3$$
$$-2 = m\,(-1)^2$$
$$m = -2$$

The equation of this graph is $y = -2\,(x - 1)^2 + 3$

Don't put −m into the equation to start, as the negative sign will come out as part of the process.

While the easiest point to use to find the value of m is the y-intercept, sometimes this is not possible and another point is needed. The process is the same.



Here the original $y = x^2$ turning point is moved across two and down three:

$$y - (-3) = m\,(x - (2))^2$$

which on cancelling negatives gives

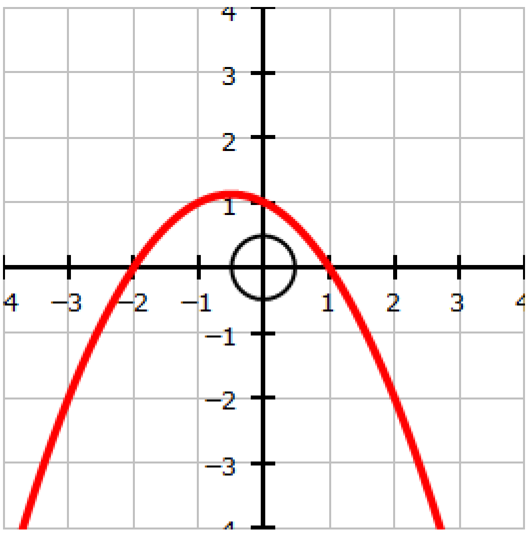
$$y = m\,(x - 2)^2 - 3$$

We can't use the y-intercept but we know that point (1, 2) is on the graph, so:

$$2 = m\,(1 - 2)^2 - 3$$
$$5 = m\,(-1)^2$$
$$m = 5$$

And so we get $y = 5\,(x - 2)^2 - 3$

Although there is always a turning point equation, sometimes the intercept method is better.



We don't know the turning point here, so:

$$y = m\,(x - 1)(x + 2)$$

The y-intercept is (0, 1) so we can substitute:

$$1 = m\,(0 - 1)(0 + 2)$$
$$1 = m\,(-1)(2)$$
$$m = -0.5$$

So the equation is: $y = -0.5\,(x - 1)(x + 2)$

The turning point equation here is the awkward $y = -0.5\,(x + 0.5)^2 + 1.25$

Practice Page 1 – Parabolas

Write equations for the following graphs. Use your graphics calculator to check them.

Use both intercept and turning point methods where possible.

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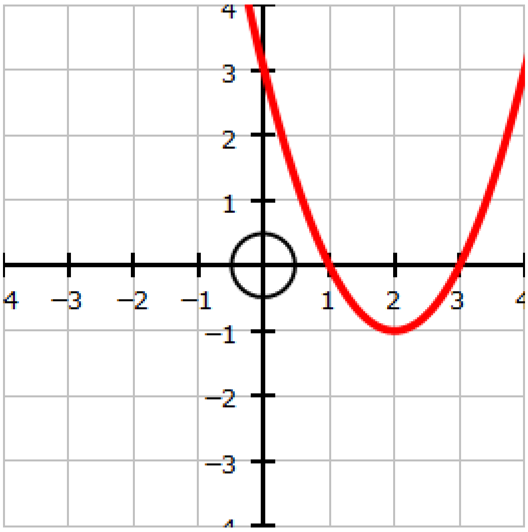

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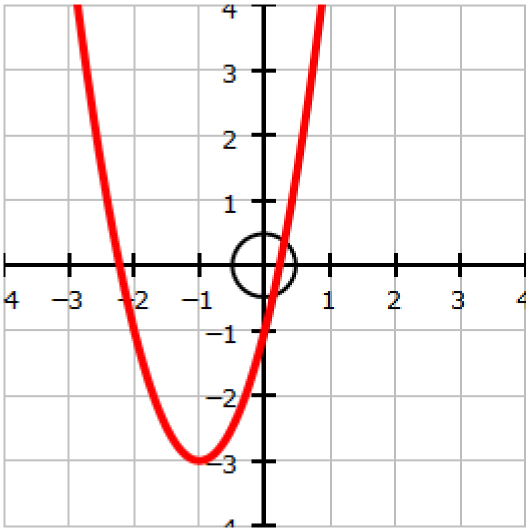

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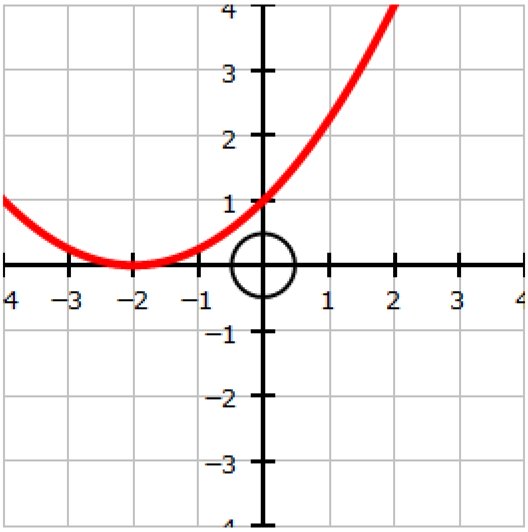

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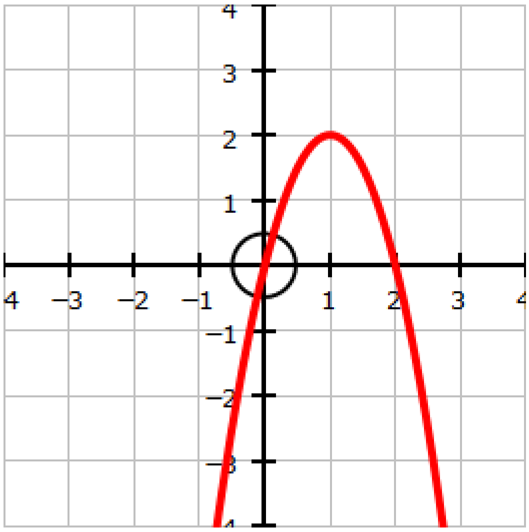

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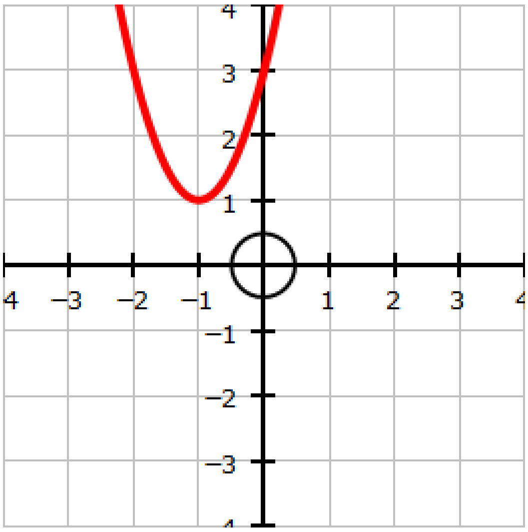

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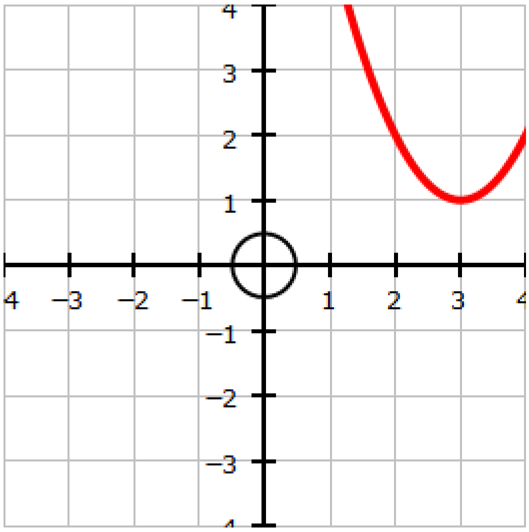

Practice Page 2 – Parabolas

Write equations for the following graphs. Use your graphics calculator to check them.

Some of these are more awkward, including fractional values.

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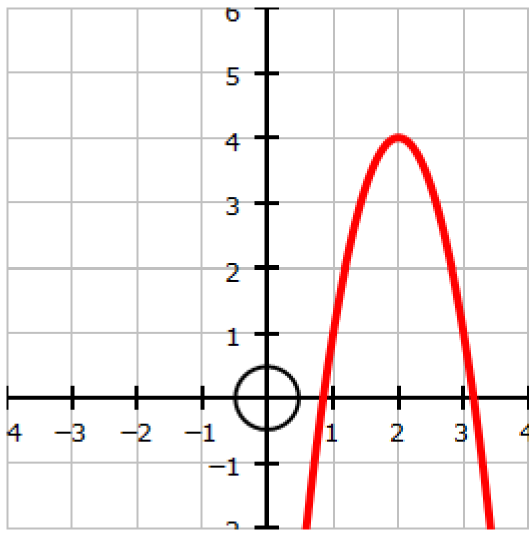

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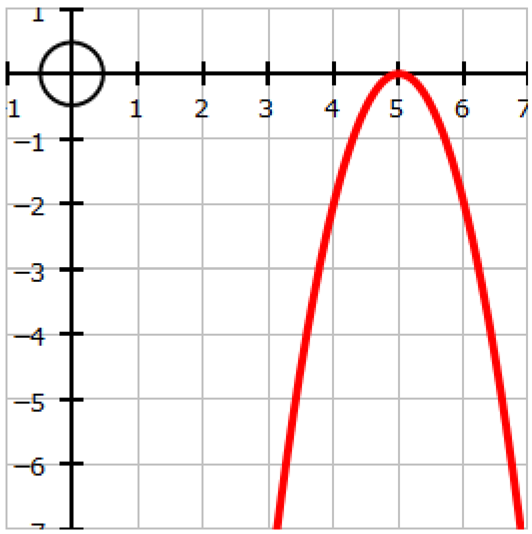

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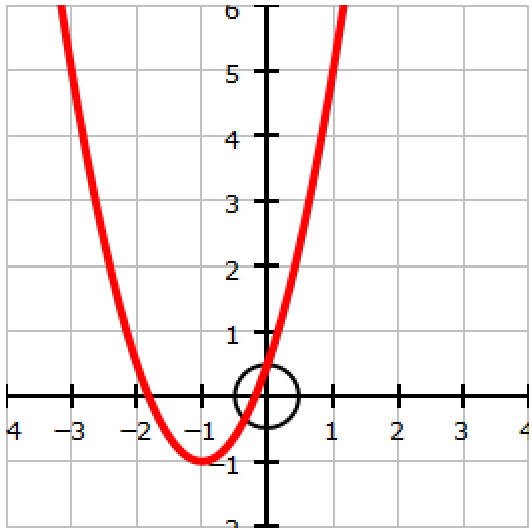

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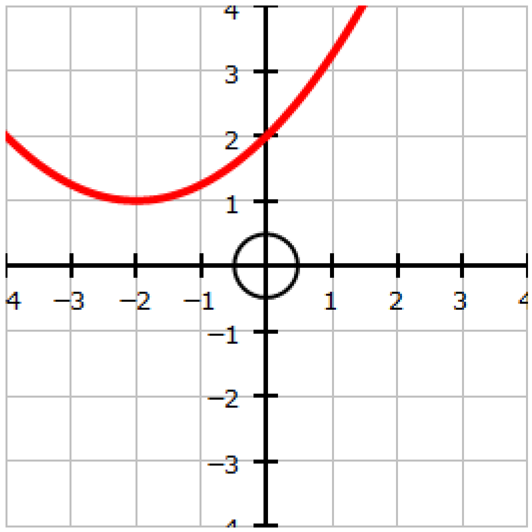

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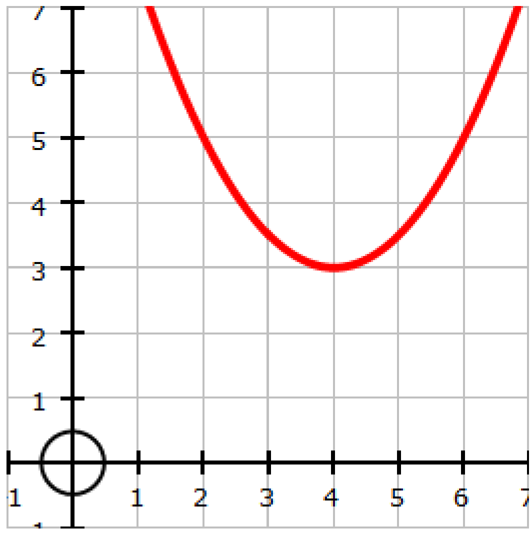

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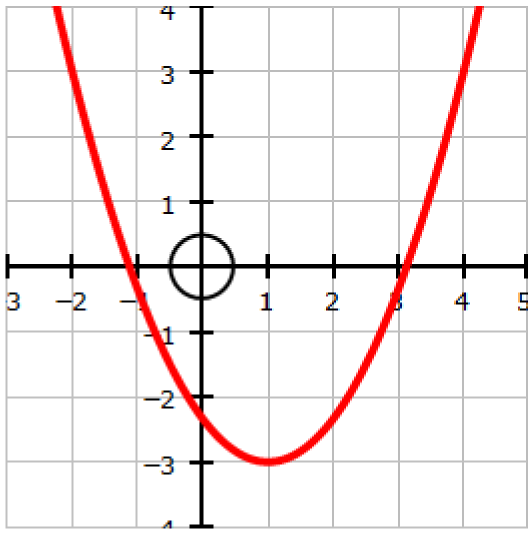


Square Root Graphs

Parabolas need not be vertical, but can be horizontal as well. There are two ways of doing this, either using the form $x = y^2$, which we do at Level 3, or the simpler $y = \sqrt{x}$ for Level 2.

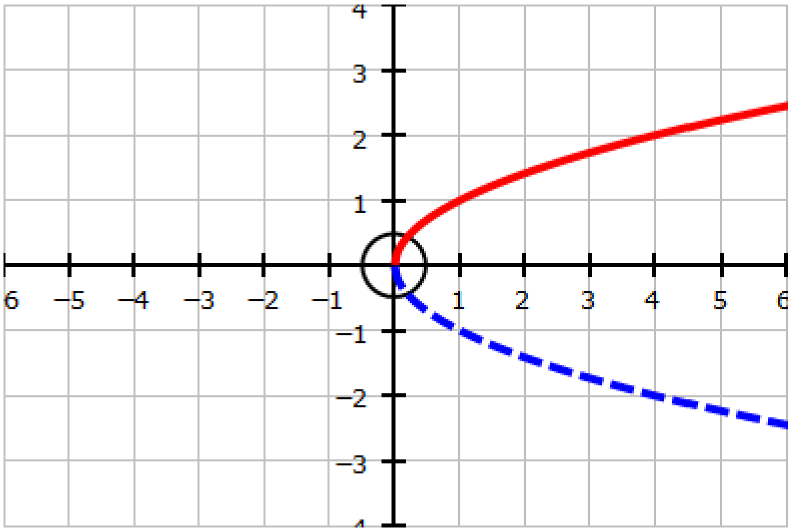
The basic graph is just a parabola on its side, where

$y = \sqrt{x}$:

is the solid line, being half a parabola, and

$y^2 = x$

is the same but adds the dotted portion to make a full parabola.



Method

All square root graphs can be written in the form: $\boldsymbol{y = m\sqrt{x - t} + c}$

where m determines the steepness of the graph;
t is the turning point's x value, still subtracted;
c is the vertical distance of the turning point in the y dimension.

1) Write out general form: $y = m\sqrt{(x - t)} + c$.
2) Use the turning point to generate t and c. Remember to apply negatives to x shifts.
3) Use a known value to calculate m.
4) Check the equation on your graphics calculator. Be careful to bracket properly.

For example the graph here has the turning point at (−2, 1), which is a shift of one up and two to the left from $y = \sqrt{x}$

Making the shifts negative we get

$y - 1 = m\sqrt{x - (-2)}$

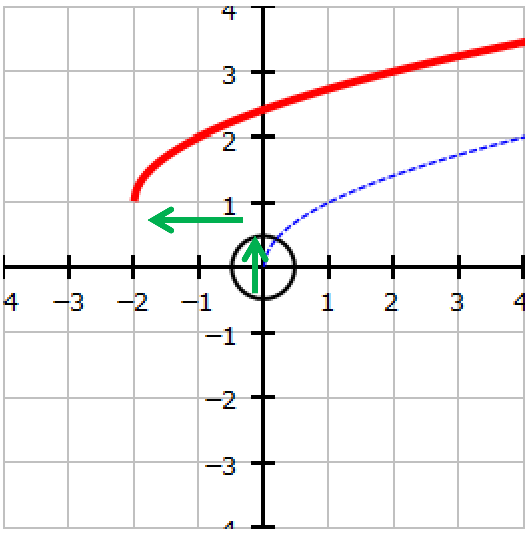
which simplifies to:

$y = m\sqrt{x + 2} + 1$

The graph goes through (2, 3) so substituting in:

$3 = m\sqrt{2 + 2} + 1$
$2 = m\sqrt{4}$
$m = 1$

and the graph is $y = \sqrt{x + 2} + 1$



Variations

The m value does not have to be positive. If negative, the parabola is flipped vertically.

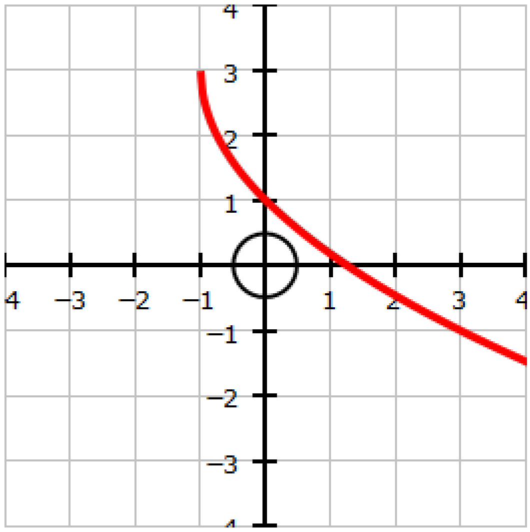
The turning point here is at (−1, 3) so

$$y = m\sqrt{x+1} + 3$$

We see (0, 1) is on the graph, so:

$$1 = m\sqrt{0+1} + 3$$
$$-2 = m\sqrt{1}$$
$$m = -2$$

The equation of this graph is $y = -2\sqrt{x+1} + 3$



Merit: The effect of increasing the m value is to increase the steepness of the graph, as with a normal parabola. However the m is in front of the x, so it is increased in the vertical direction.

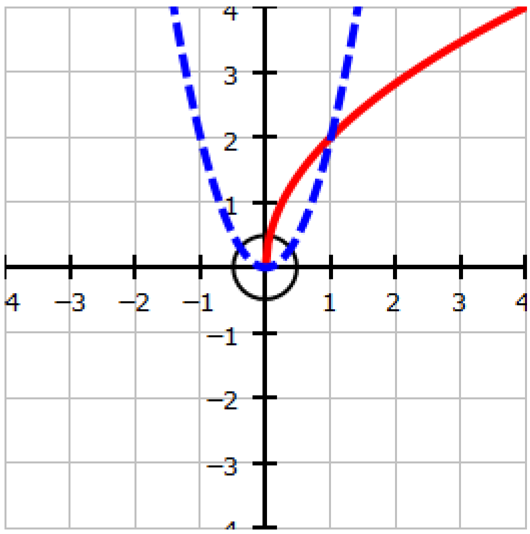
That means the same numerical multiplier m gives quite different resulting changes to the shape.

$y = 2\sqrt{x}$ is a "fatter" parabola than $y = \sqrt{x}$.

But $y = 2x^2$ is a "skinnier" parabola than $y = x^2$.

Both are twice as steep, in the vertical dimension.

It pays to never calculate the m value by eye, since the result is counter-intuitive compared to a normal parabola.



Excellence: The graph can be flipped in a horizontal direction by replacing x with $-x$ but this will rarely be required.

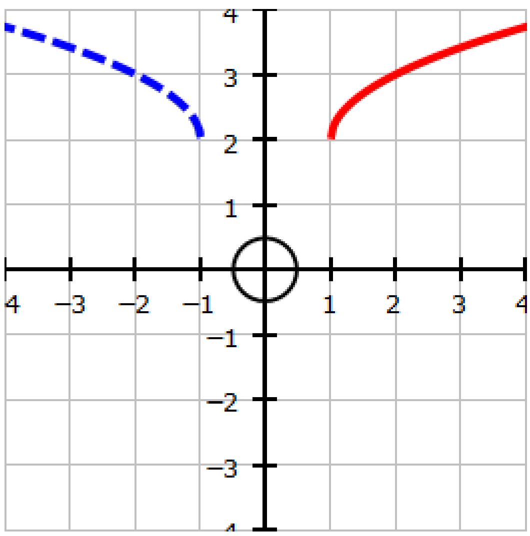
The graphs shown are:

$$y = \sqrt{x-1} + 2$$

and

$$y = \sqrt{-x+1} + 2$$

Note: to reflect a graph in the y-axis the replacement of x with −x also needs to be mirrored in any x shift, which must also be changed the negative of the positive version.



Practice Page 3 – Square Root Graphs

Write equations for the following graphs. Use your graphics calculator to check them.

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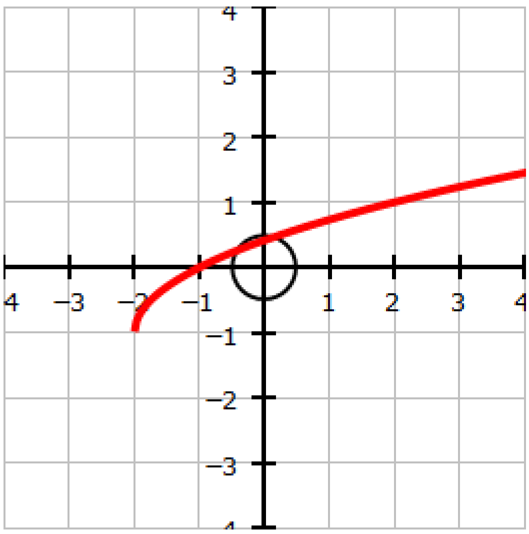

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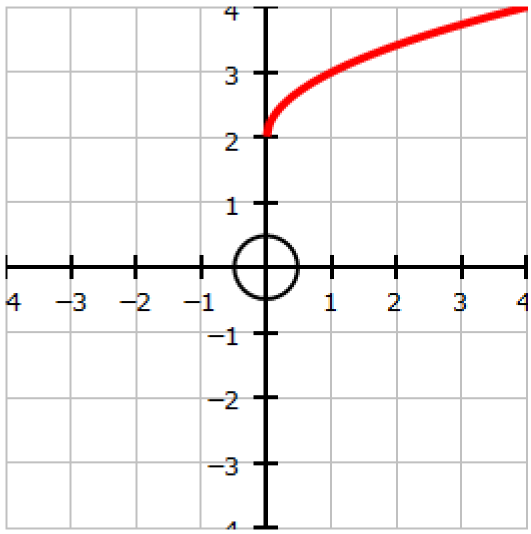

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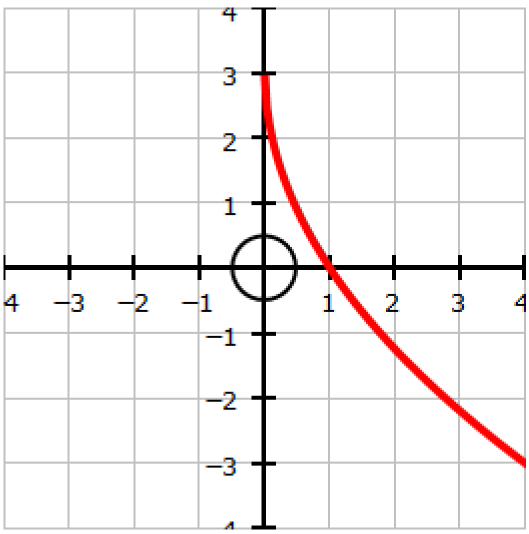

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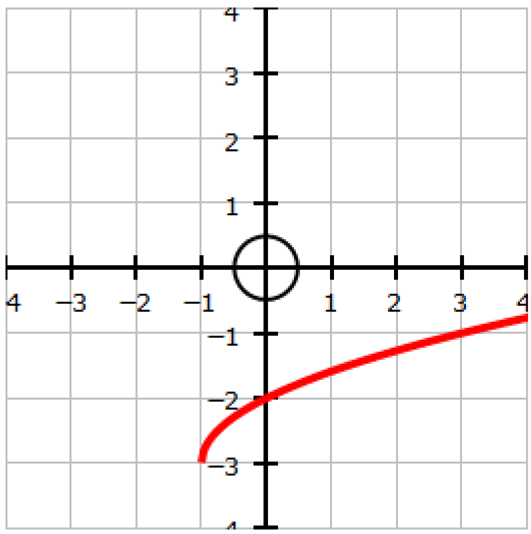

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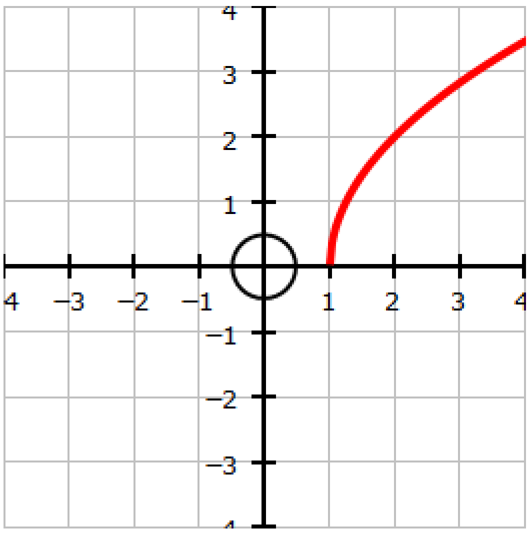

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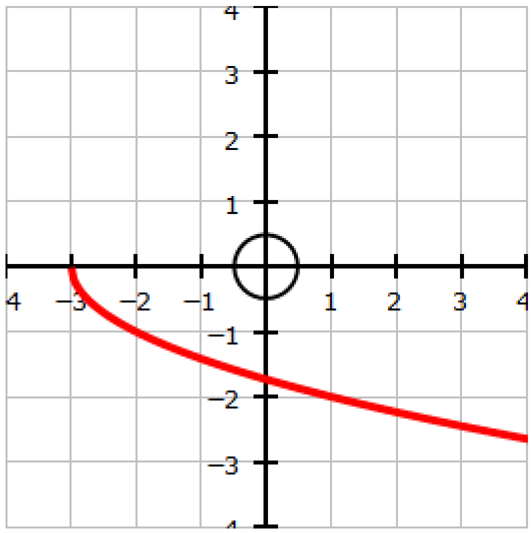

Practice Page 4 – Square Root Graphs

Harder graphs. Use your graphics calculator to check them.

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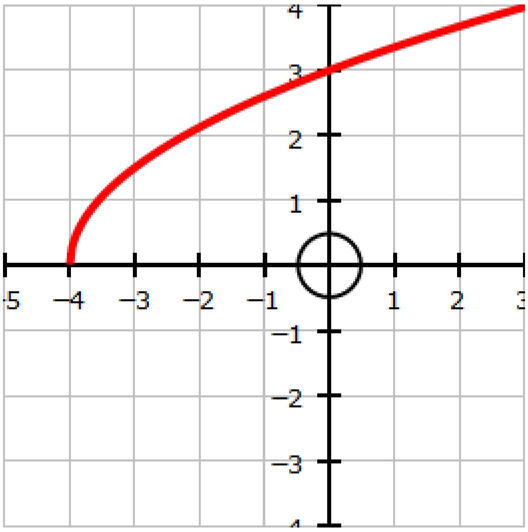

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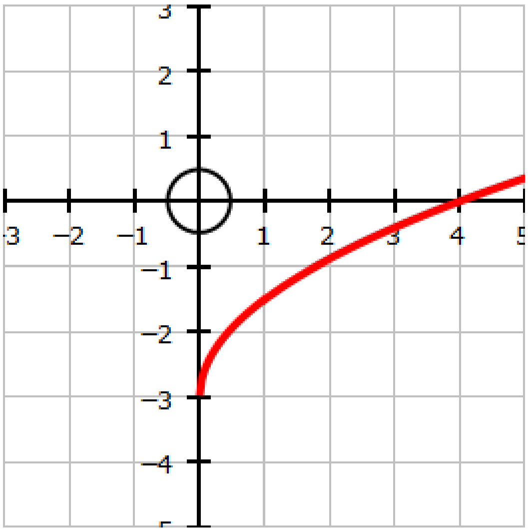

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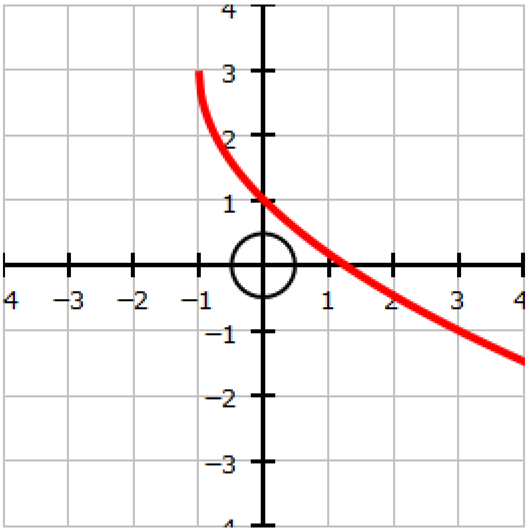

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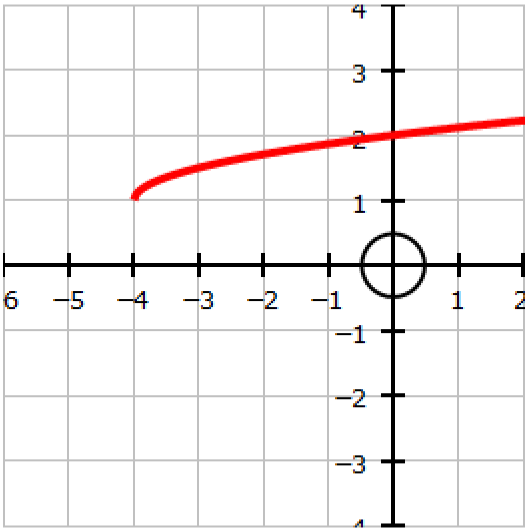

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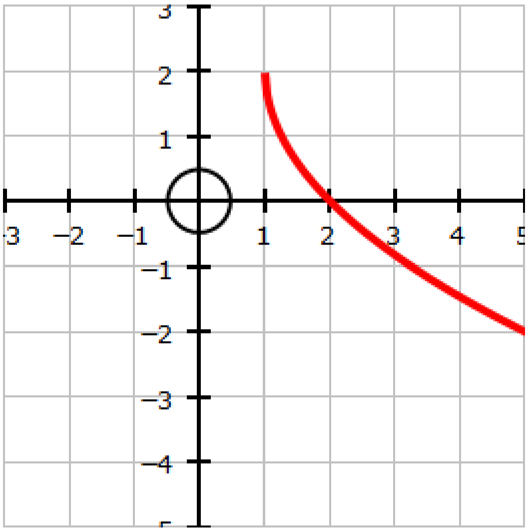

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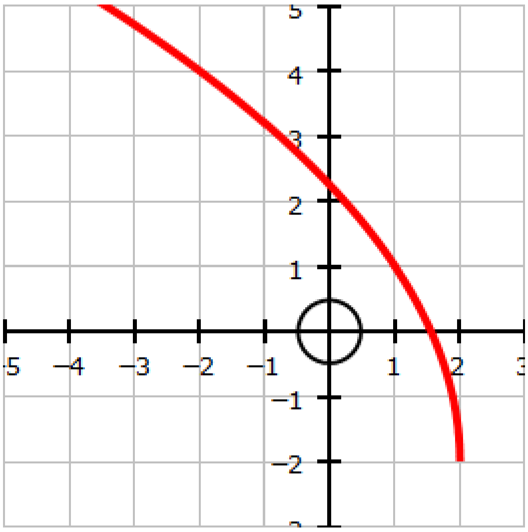
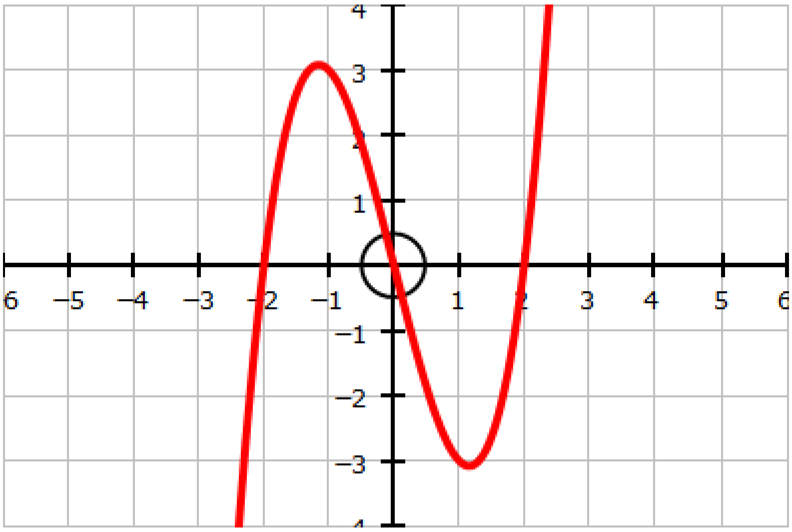


Cubic Graphs

Cubic polynomials are of the general form $y = y = ax^3 + bx^2 + cx + d$, where a, b, c and d = are any number. At Level 2 we only deal with those that factorise to whole numbers (and so have simple intercepts).

The graph shown is

$y = (x - 2)x(x + 2)$



The turning point areas are not symmetrically placed between intercepts – here they are ***not*** *at $x = -1$ and $x = 1$ and the tops are not symmetric like parabolas.*

Instead cubics have rotational symmetry.

Method

The general form for cubics is $y = ax^3 + bx^2 + cx + d$,

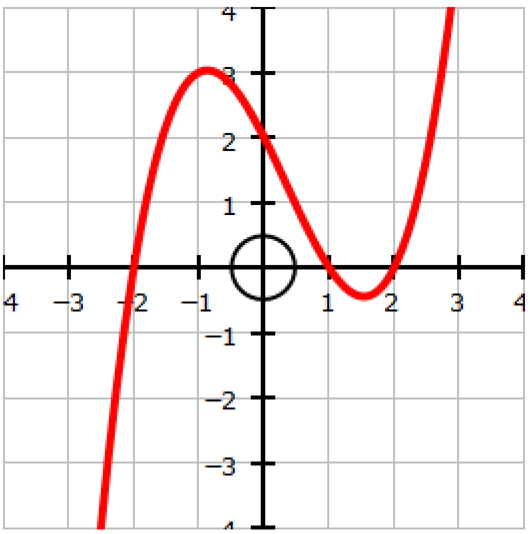
The intercept form is more useful when the intercepts are known: $\mathbf{y = m\,(x - a)(x - b)(x - c)}$

where m determines the steepness of the graph;
a, b and c are the x-intercepts. Note the negative signs, same as parabolas.

1) Use the intercepts to write in the form: $y = m\,(x - a)(x - b)(x - c)$.
2) Use a known value to calculate m.
3) Check the equation on your graphics calculator.

The y-intercept is the best value to use to calculate m, if you have it as it makes the calculations easiest. Even though they sometimes look like it, turning points are never on whole number values if the intercepts are, so should not be used to calculate m.

For example the graph here has x-intercepts at (–2, 0), (1, 0) and (2, 0)



Using the x-intercepts as negatives we get

$y = m\,(x + 2)(x - 1)(x - 2)$

The graph goes through (0, 2) so substituting in:

$2 = m\,(0 + 2)(0 - 1)(0 - 2)$
$2 = m\,(2)(-1)(-2)$
$m = 0.5$

and the graph is $y = 0.5\,(x + 2)(x - 1)(x - 2)$

Variations

If the m value is negative then the graph is flipped along the x-axis compared to the positive.

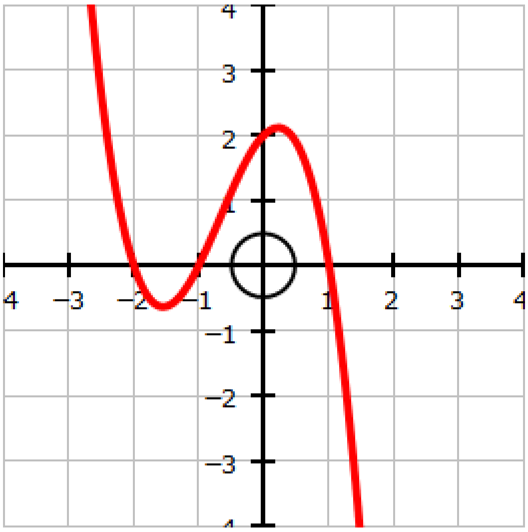
The graph here shows

$$y = -(x-1)(x+1)(x+2)$$

When calculating a m value known to be negative, just leave the m as a positive value and the negative sign will come out naturally from the working of the known point.

We can also write this as $y = (1-x)(x+1)(x+2)$

because $-(x-a) = (a-x)$



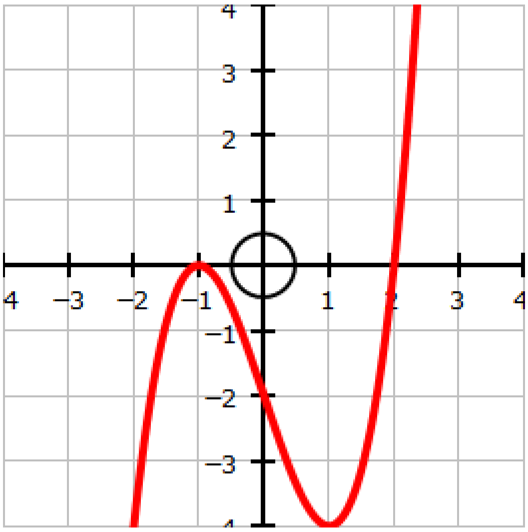
The graph can turn on an intercept. In that case the intercept counts twice, like parabolas.

This is the graph of

$$y = (x+1)^2(x-1)$$

It can, of course, be written with the bracket repeated:

$$y = (x+1)(x+1)(x-1)$$



Excellence: Cubics need only have one intercept. Sometimes the formula for such ones can be determined by taking a line *as if it is the x-axis* and then correcting for it afterwards.

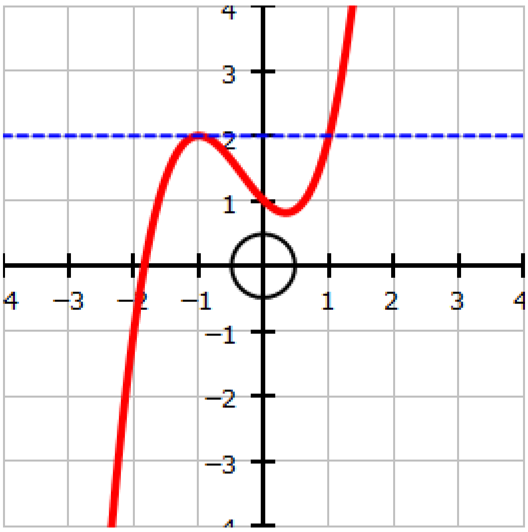
The graph only has one intercept, but if we imagine the line $y = 2$ was the x-axis, shown dotted, then it would have the equation:

$$y = (x+1)^2(x-1)$$

and then we correct for the "wrong" axis by adding a vertical shift, as we would for a parabola, giving the correct formula of:

$$y = (x+1)^2(x-1) + 2$$

This expands to $y = x^3 + x^2 - x + 1$ *in the general form.*



Practice Page 5 – Cubic Graphs

Write equations for the following graphs. Use your graphics calculator to check them.

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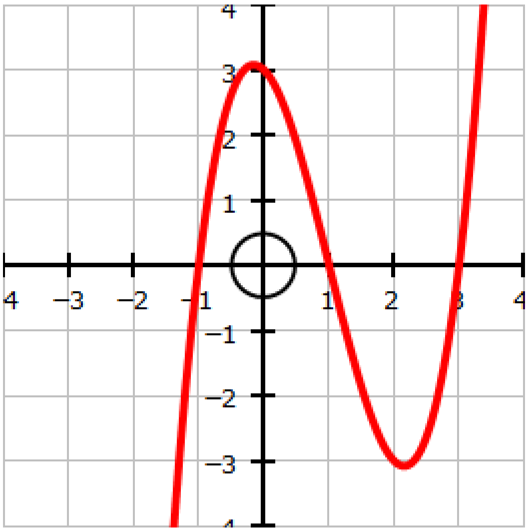

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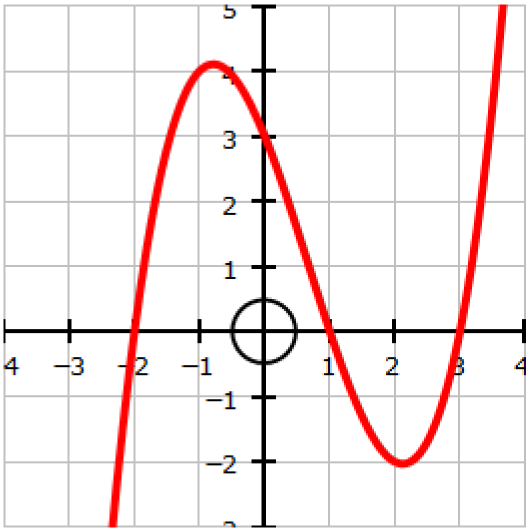

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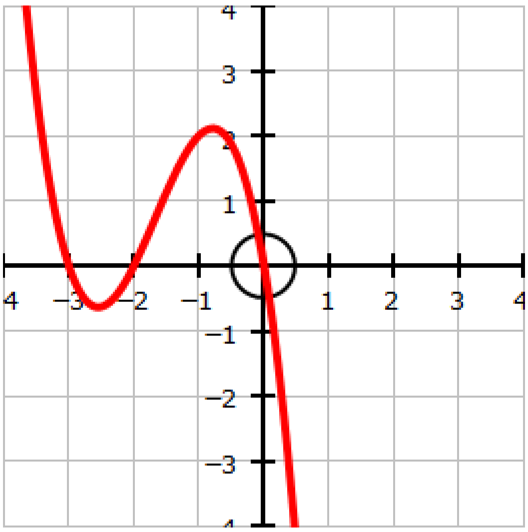

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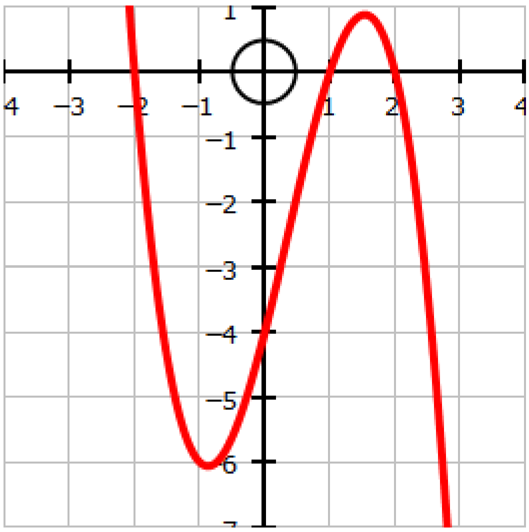

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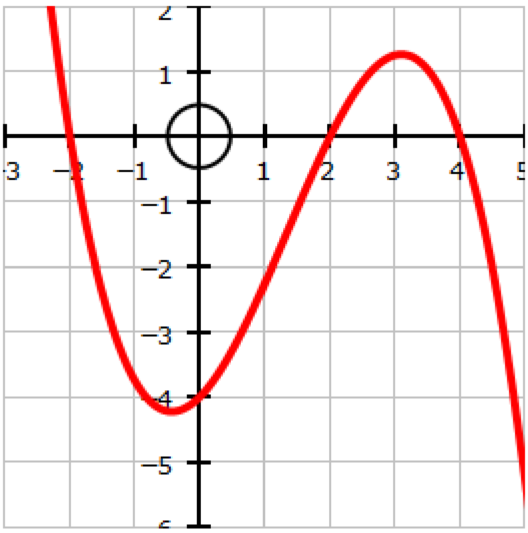

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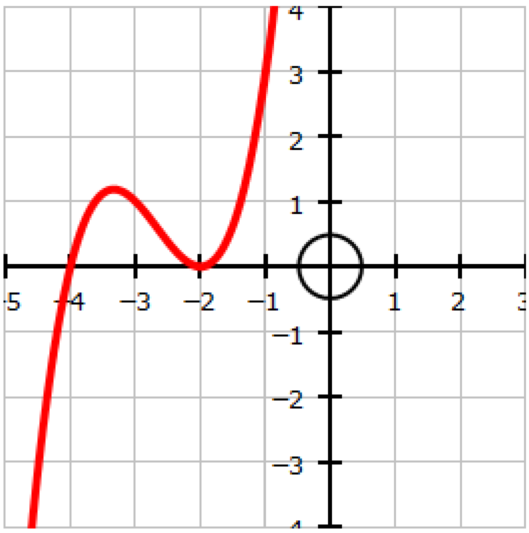
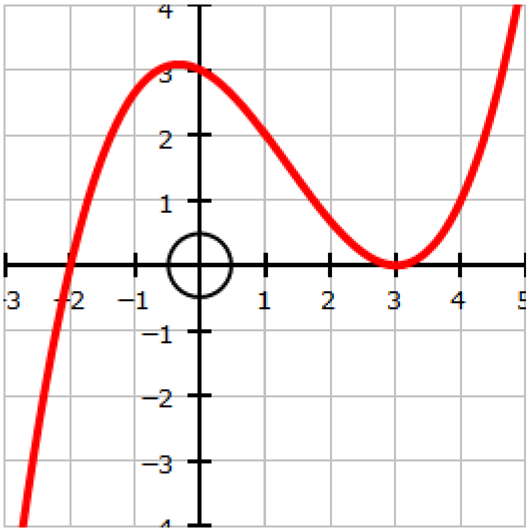


Practice Page 6 – Cubic Graphs

Harder graphs. Use your graphics calculator to check them.

31



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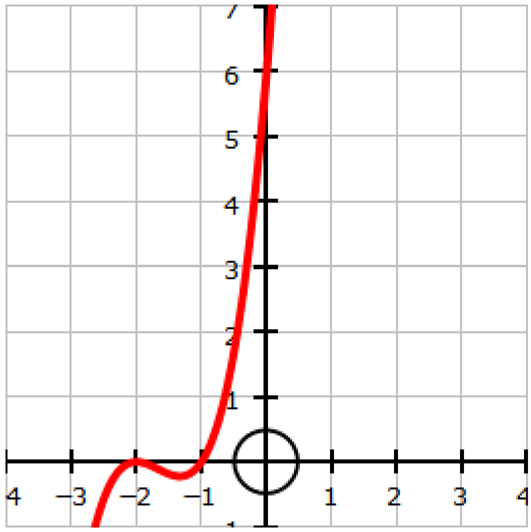

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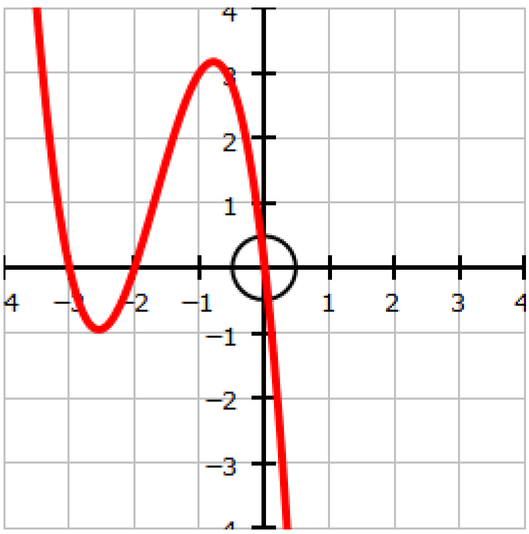

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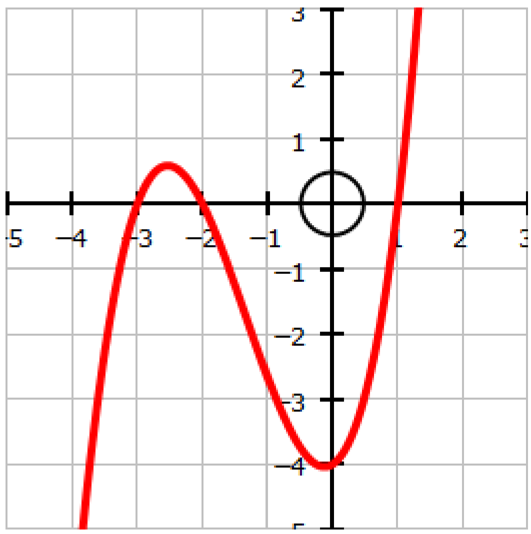

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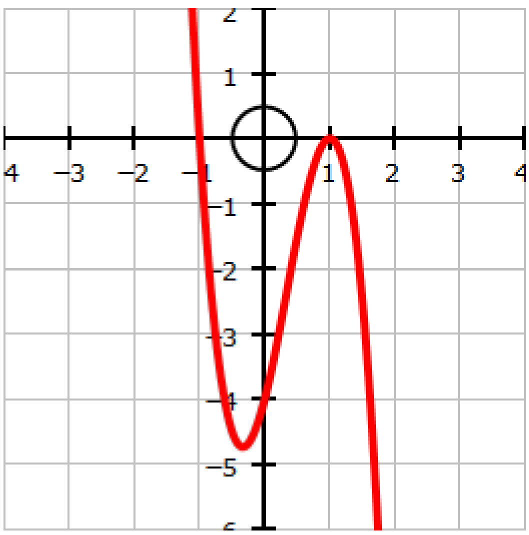

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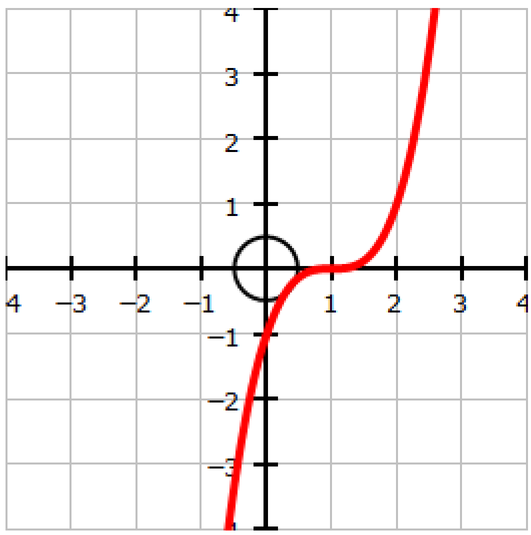


Hyperbolas

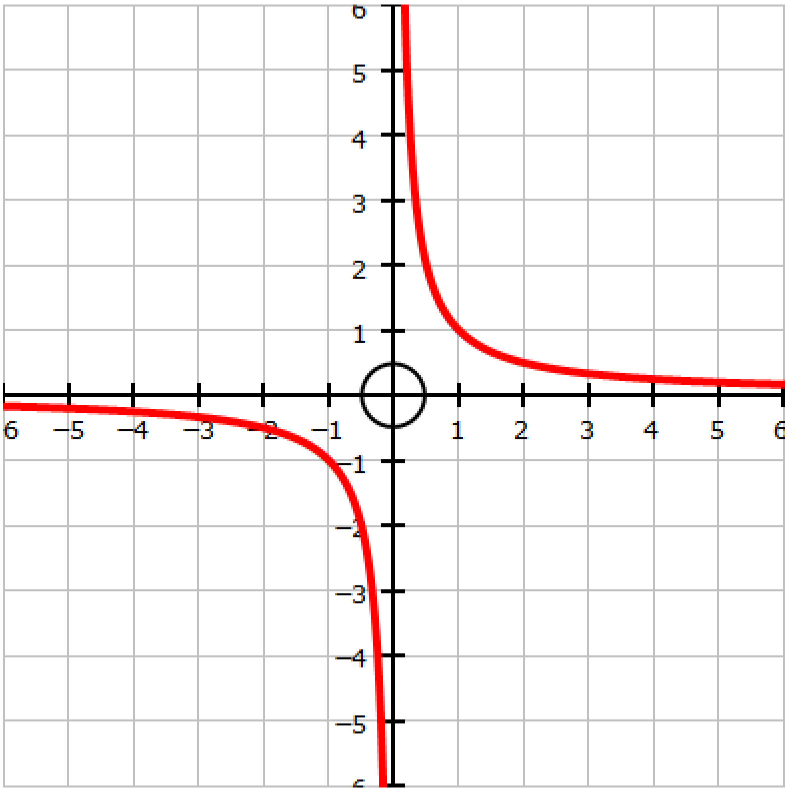
Hyperbolas are of the general form $y = \frac{1}{x}$ (which can also be written as $xy = 1$)

The graph shown is

$$y = \frac{1}{x}$$

The graph is fully symmetric, with rotational symmetry of 180° as well as mirror lines along the diagonals y = x and y = −x

All hyperbola have this symmetry.



The graph has no value for the y-axis (which is the line $x = 0$) as division by zero gives no result. The result is an asymptote that gets infinitely close from both sides.

There is also a similar horizontal asymptote on the x-axis (i.e. $y = 0$) because there is no number x such that $0 = 1/x$

An **asymptote** is a straight line that continually approaches a curve but never touches it.

Method

Write the graph in the general form: $$\boldsymbol{y = \frac{m}{x-a} + c}$$

where m determines the steepness of the graph;
a is the distance the vertical asymptote is shifted. Note the negative.
c is the distance the horizontal asymptote is shifted.

1) Use the x and y shifts of the asymptotes to write in the form: $y = \frac{m}{x-a} + c$
2) Use a known value to calculate m. Use an intercept, if you know one.
3) Check the result on your graphics calculator. The denominator must be bracketed.

Note that the x shift is negative on the x part of the equation, as normal, and the y shift is positive because we don't write it beside the y but instead on the other side of the equation.

It can help to think of the equation as: $(x - a)(y - b) = m$ to explain the shifts.

Variations

The x and y shifts can be done either by how the asymptotes move, or from the centre of rotation (which is where the asymptotes cross).

The thin dotted line is that of $y = \frac{1}{x}$.

The solid graph is centred on (2, 1), so

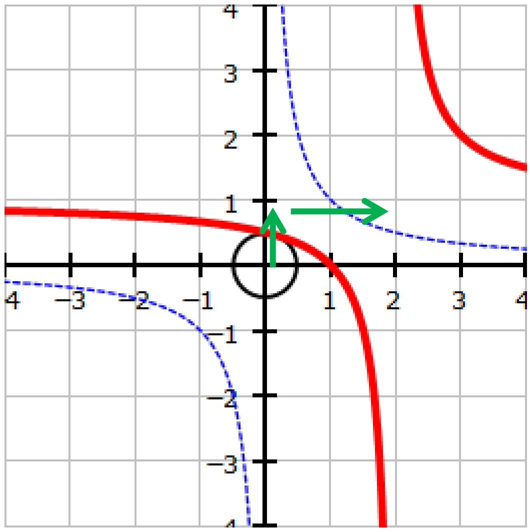
$$y = \frac{m}{x-2} + 1$$

The point (3, 2) is on the graph

$$2 = \frac{m}{3-2} + 1$$

$$m = 1$$

So the equation is: $y = \frac{1}{x-2} + 1$



If the m value acts as multiplier for the distance from the x-axis, as with all the graphs, but visually it works to look as if it is expanding the graph out from the origin.

The dotted line is that of $y = \frac{1}{x}$

The graph is still centred on (0, 0) so there are no x or y shifts, only a multiplier:

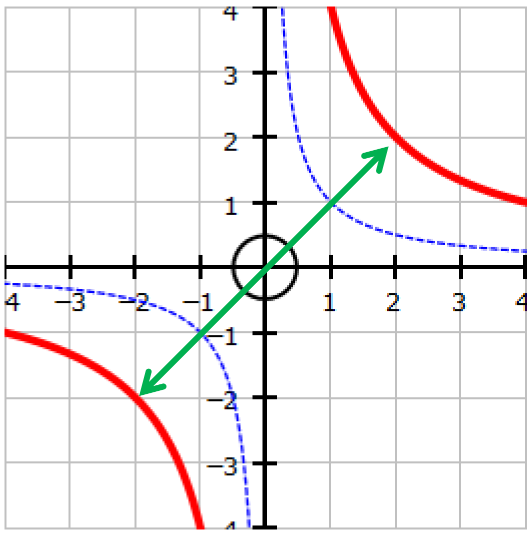
$$y = \frac{m}{x-0} + 0$$

The point (2, 2) is on the graph

$$2 = \frac{m}{2}$$

$$m = 4$$

So the equation here is: $y = \frac{4}{x}$



Negative m values reverse the corners the graph occupies, by reflecting in the x-axis.

The dotted line is that of $y = \frac{1}{x}$

The graph is still centred on (0, 0) so there are no x or y shifts, only a multiplier:

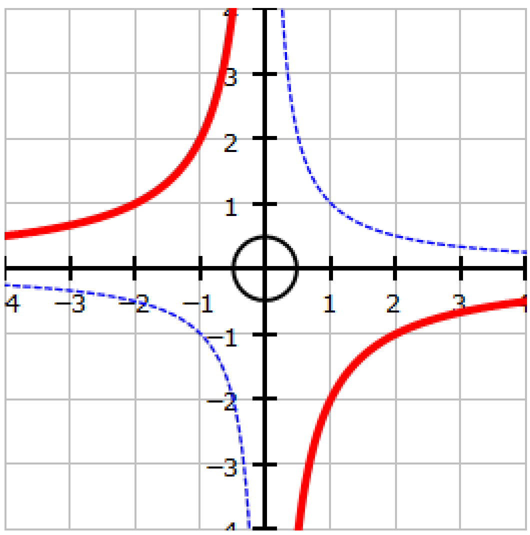
$$y = \frac{m}{x-0} + 0$$

The point (2, −1) is on the graph

$$-1 = \frac{m}{2}$$

$$m = -2$$

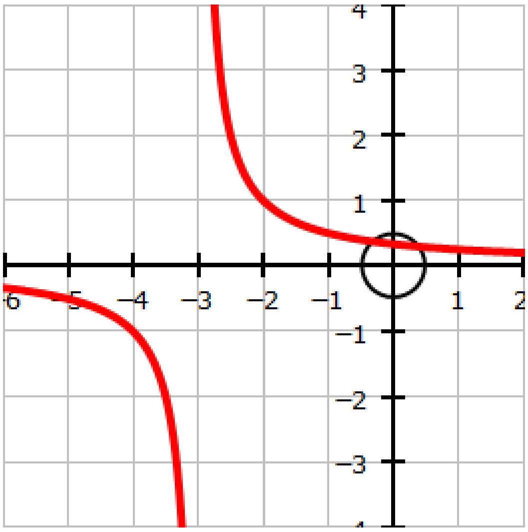
So the equation is: $y = \frac{-2}{x}$



Practice Page 7 – Hyperbolas

Write equations for the following graphs. Use your graphics calculator to check them.

37



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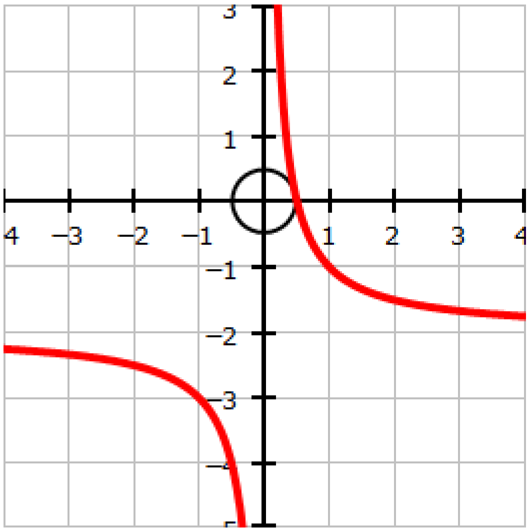

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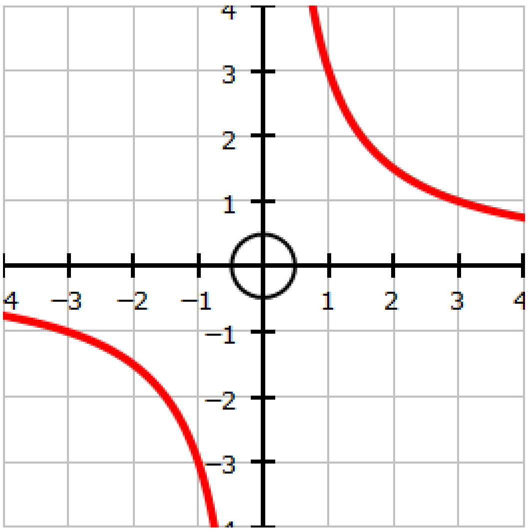

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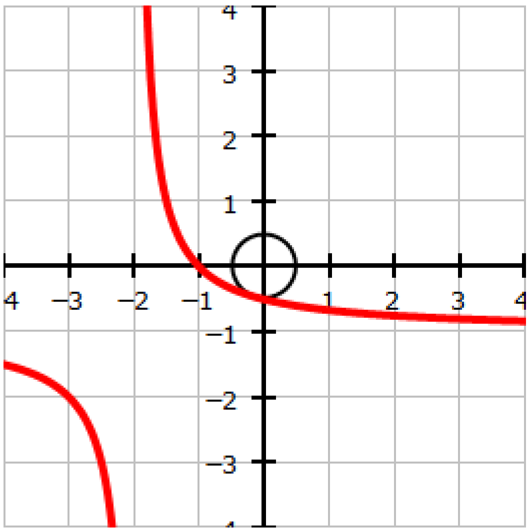

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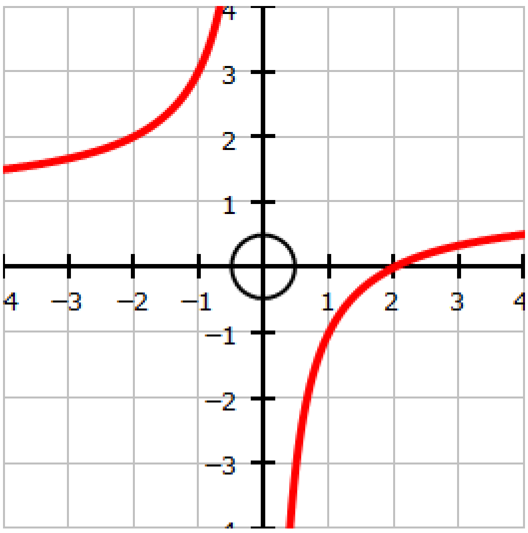

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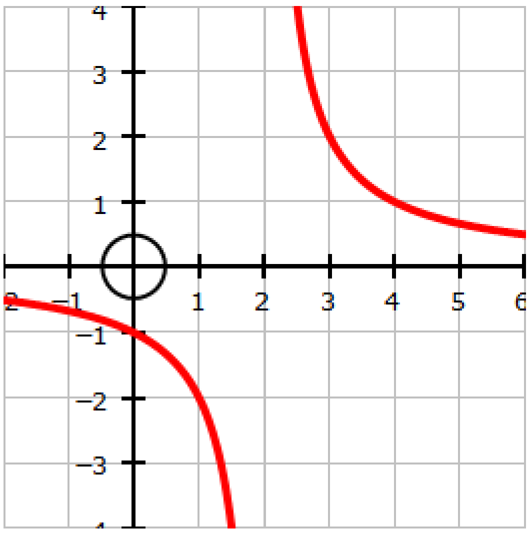
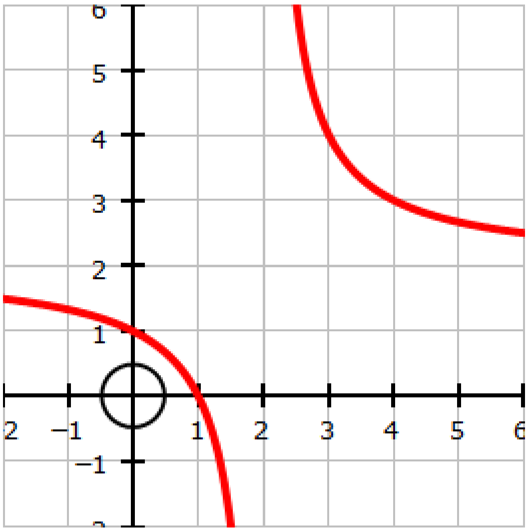


Practice Page 8 – Hyperbolas

Some harder equations.

43


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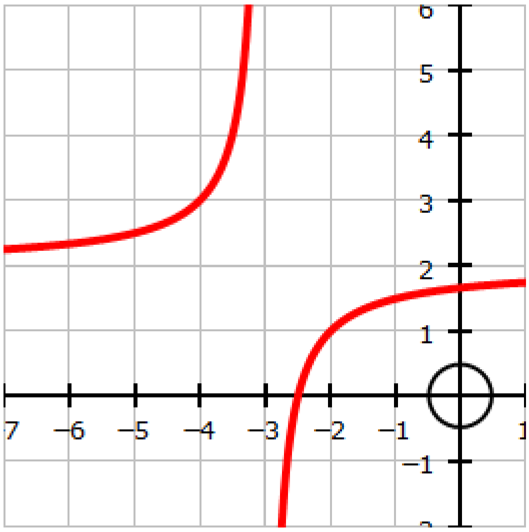

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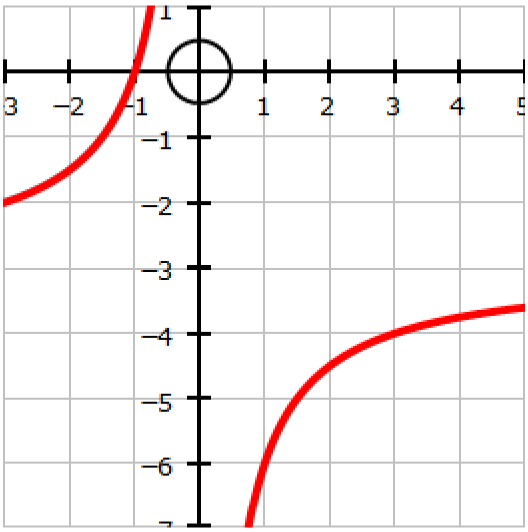

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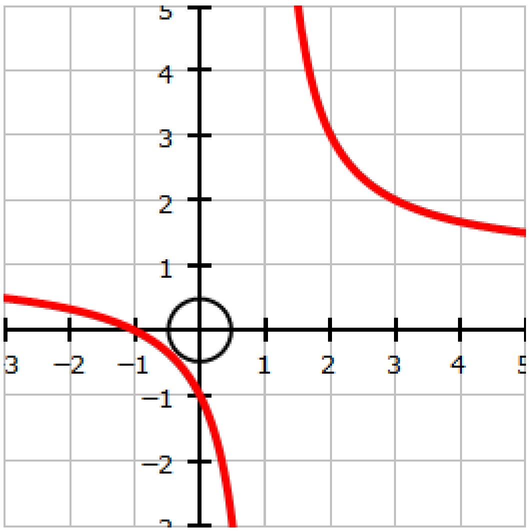

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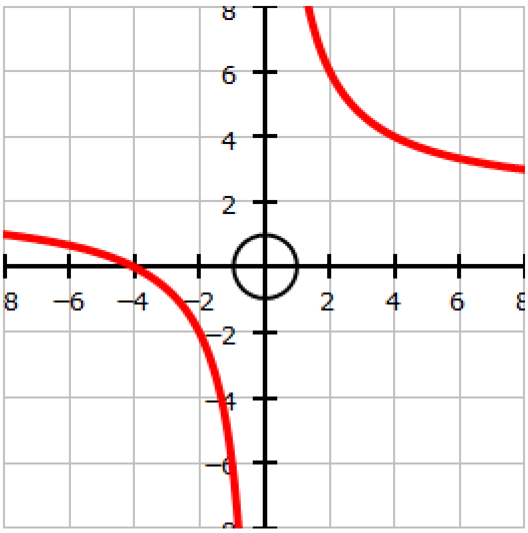
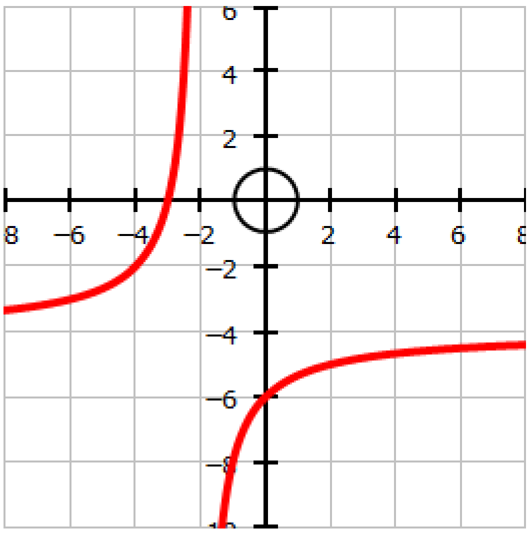


Note: this is a different scale

48


Note: this is a different scale

Absolute Value Graphs

The absolute value of a number is its value without any negative sign, which we indicate with vertical lines either side of the expression.

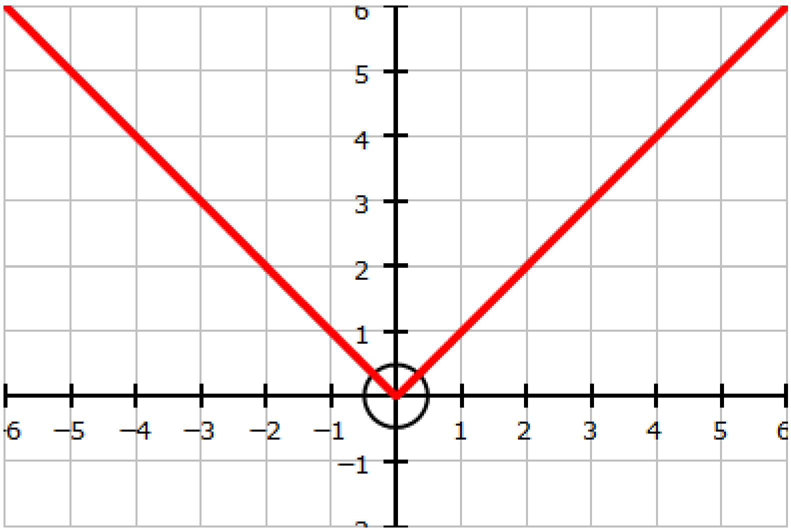
That means that $|\,3\,| = 3$ and $|\,-3\,| = 3$

The graph shown is

$y = |\,x\,|$

All absolute value graphs have vertical mirror symmetry.

Their turning point is generally called the vertex.



Method

We write the graph in the general form: $\boldsymbol{y = m\,|\,x - a\,| + c}$

where m determines the steepness of the graph, in the same way as a line;
a is the horizontal distance the vertex is shifted from (0, 0). Note the negative.
c is the vertical distance the vertex is shifted from (0, 0).

1) Use the x and y shifts of the vertex to write in the form: $y = m\,|\,x - a\,| + c$

2) Use a known value to calculate m, or find gradient as you would for a line.

   Note that you need to calculate m using values on the right side of the vertex only.

3) Check the result on your graphics calculator.

   The absolute value function in the graph menu is found using the "Optn" (option) button then "Num" (number) then "Abs" (absolute value). You need to always bracket the absolute part $|\,x - 3\,|$ = "Abs(x−3)".

Example

The vertex has been moved two places left and one down

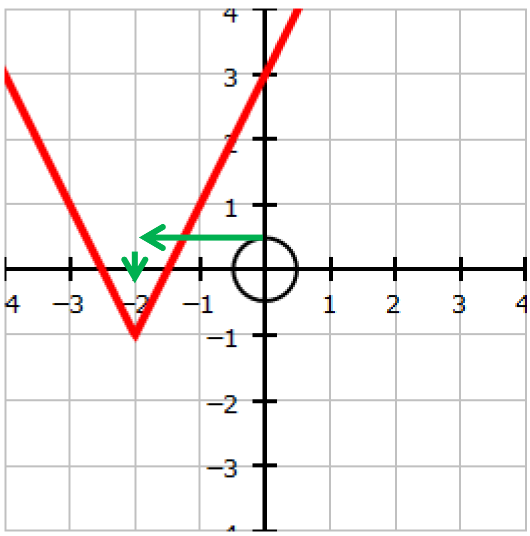
$y = m\,|\,x - 2\,| - 1$

Note the x shift is negative.

The gradient on the right hand side is 2, so the equation is:

$y = 2\,|\,x - 2\,| - 1$

A check on the graphics calculator for y = Abs($x - 2$) − 1 *shows that this is correct.*



Variations

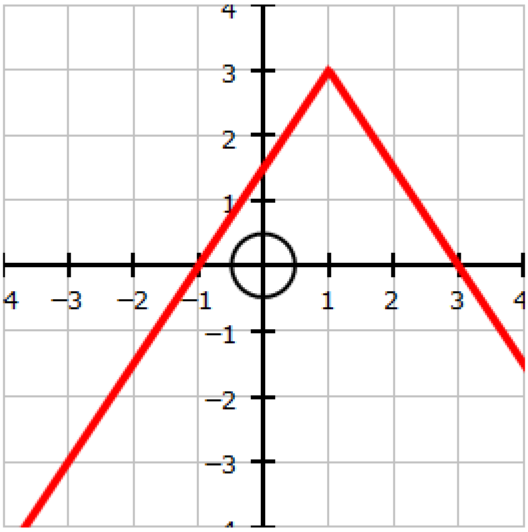
The m value does not have to be positive. If negative, the graph is upside down.

The vertex here is at (1, 3) so

$$y = m\,|\,x - 1\,| + 3$$

The gradient is down to the right side, so the correct equation is:

$$y = -1.5\,|\,x - 1\,| + 3$$

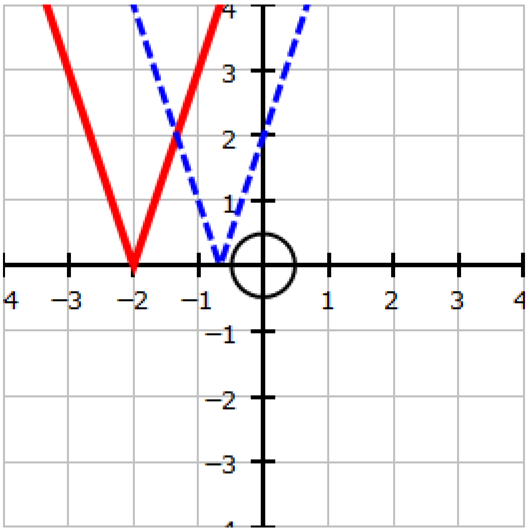


Note that although you can place the multiplier inside the absolute value brackets, the results are different, and less predictable, from placing it outside. Only place them outside.

The solid line is $y = 3\,|\,x + 2\,|$

The dotted line is $y = |\,3x + 2\,|$

The difference is caused by how the 3 of the 3x affects the + 2 as well as the x.



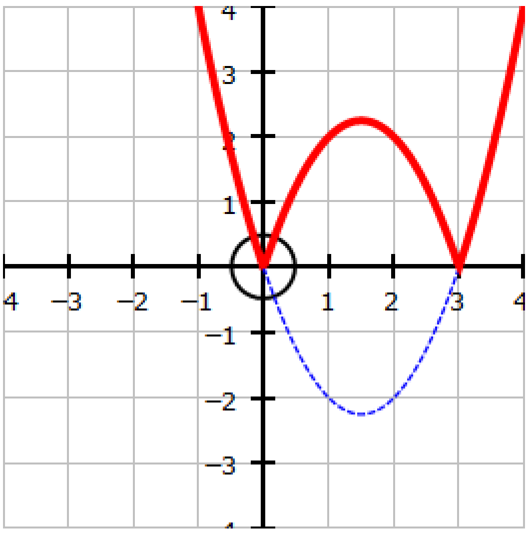
Excellence: Any graph can be made absolute value, not just lines. The result is to replace negative y values with positive ones (before any y shift).

The solid line is:

$$y = |\,x\,(x - 3)\,|$$

which turns the negative parts of $y = x\,(x - 3)$, shown dotted, into positive values.

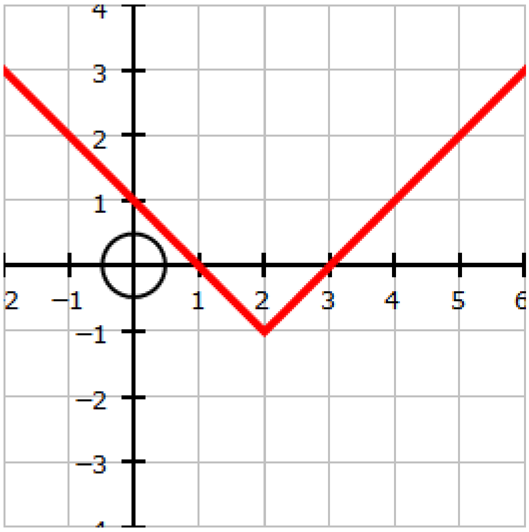
This is not required at Level 2.



Practice Page 9 – Absolute Value Graphs

Write equations for the following graphs. Use your graphics calculator to check them.

49



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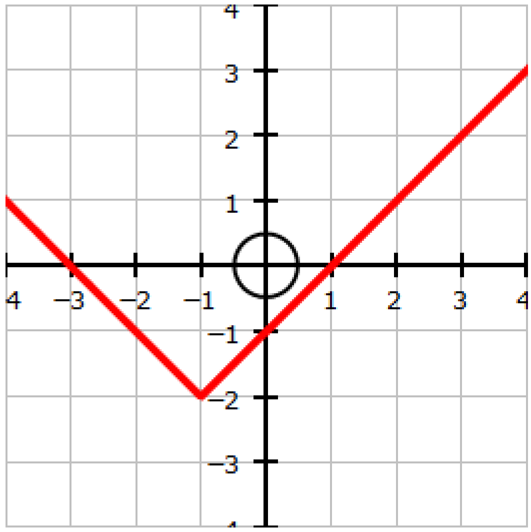

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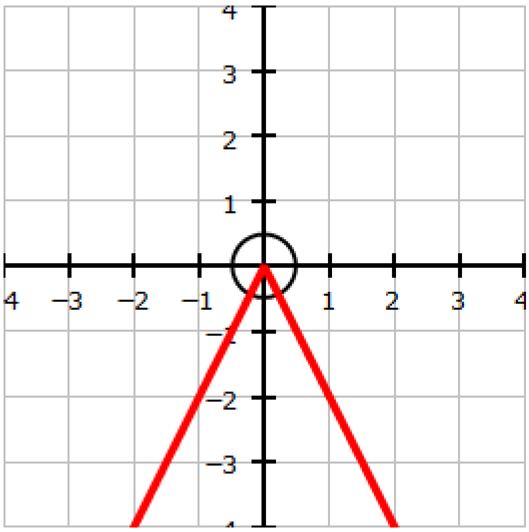

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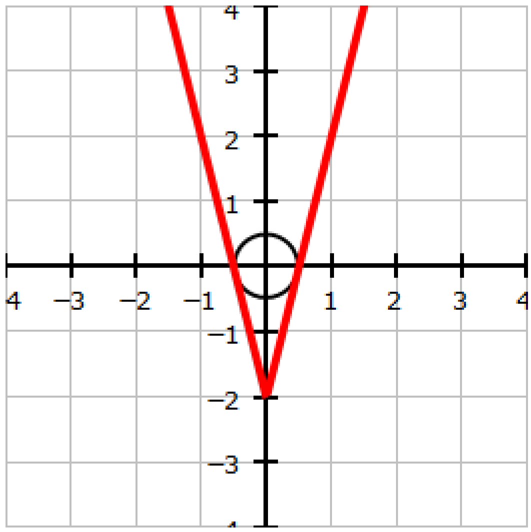

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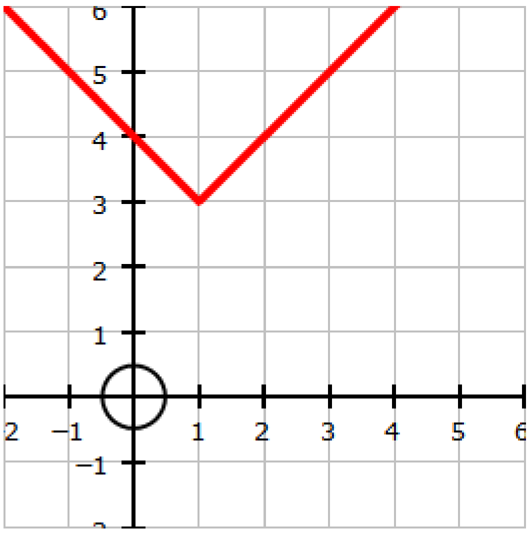

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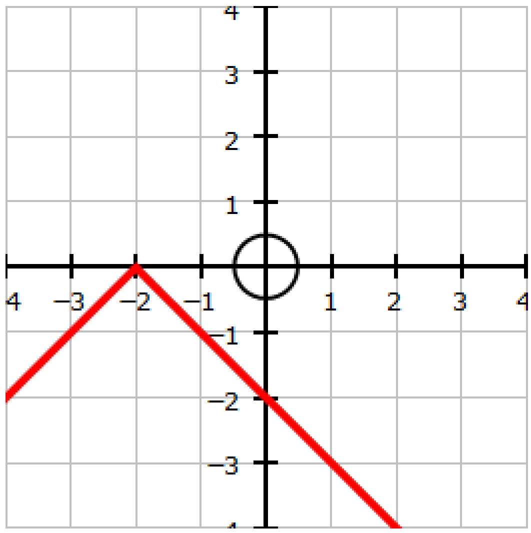
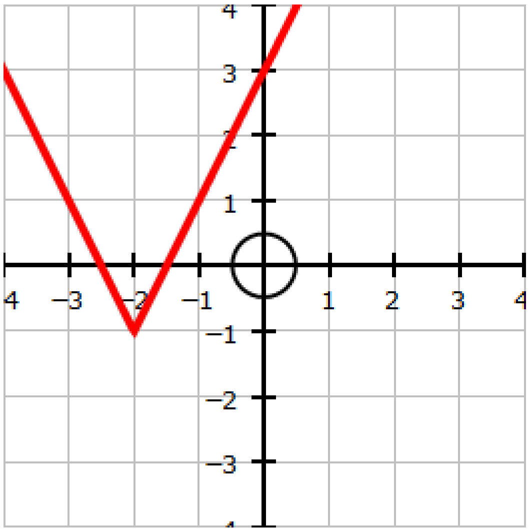


Practice Page 10 – Absolute Value Graphs

Some more equations.

55



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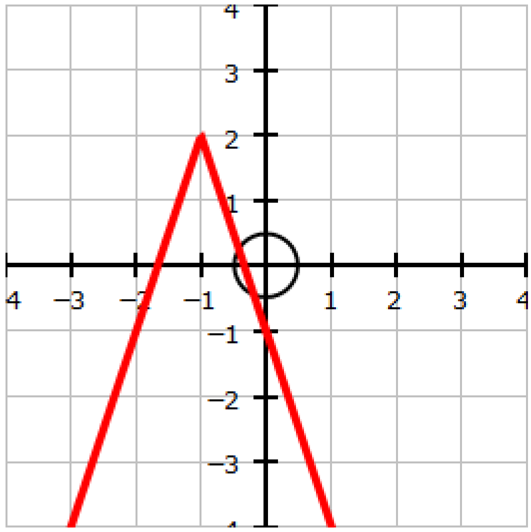

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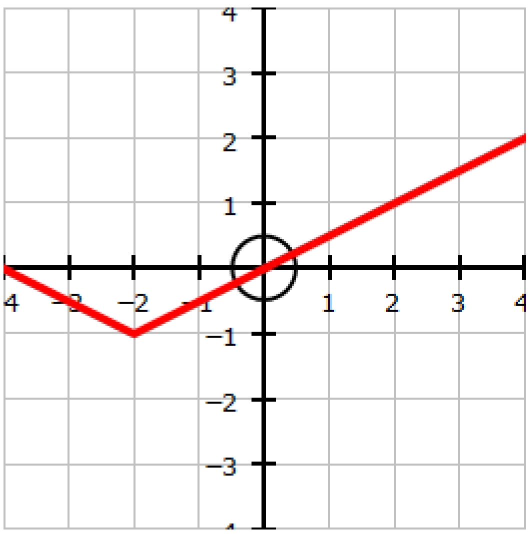

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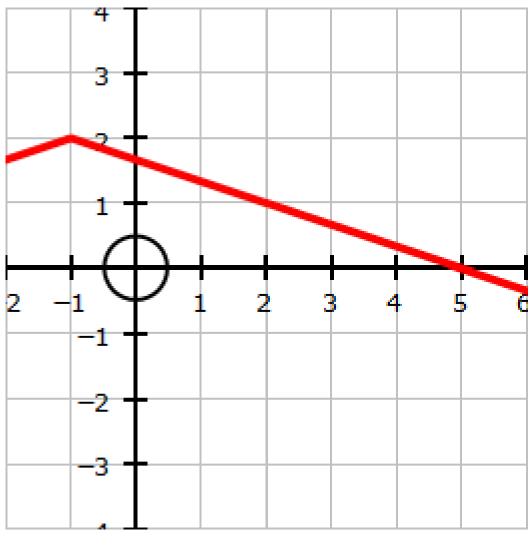

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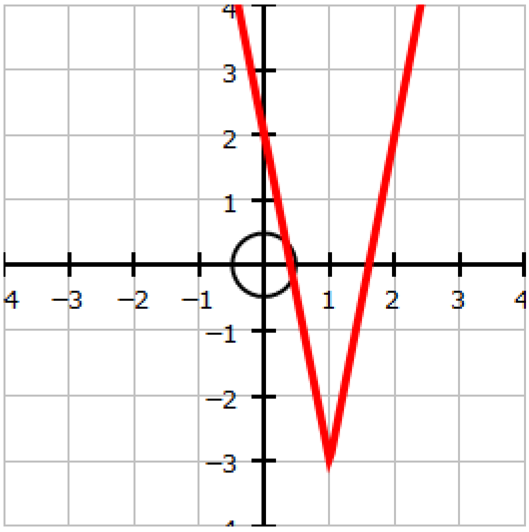

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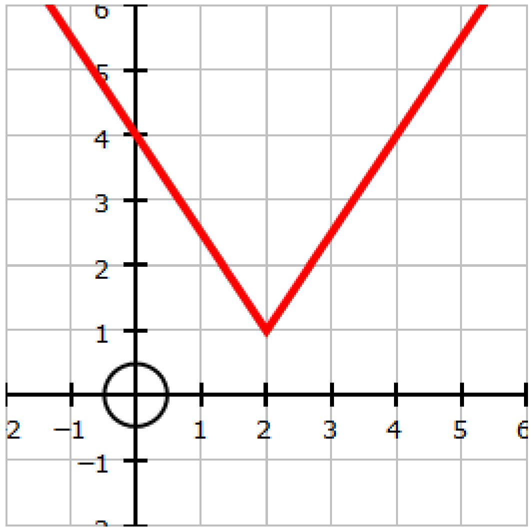


Exponential Graphs

Exponential growth curves are common in nature and economics and a general understanding of them is useful. Unfortunately, their equations are difficult to write.

The graph shown is

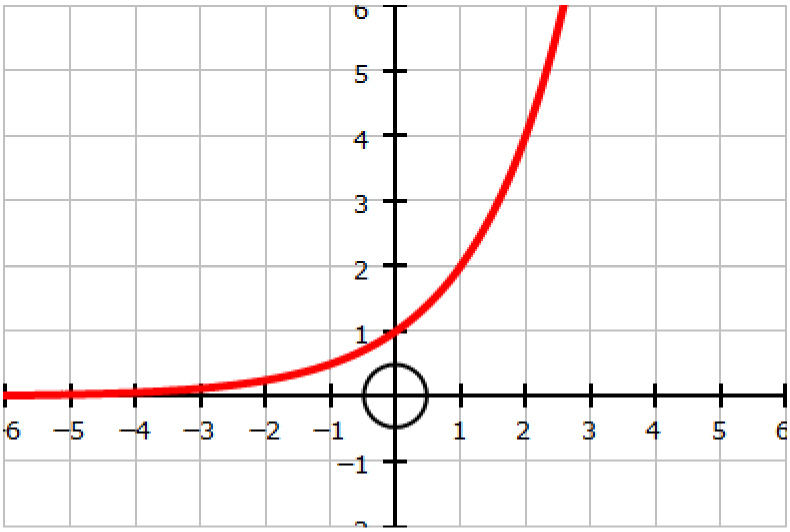
$y = 2^x$

The "base" of 2 means the graph doubles in height for every one across.

Because $b^0 = 1$ for all values of b, the point is (0, 1) is always the y-intercept for $y = b^x$

There is a horizontal asymptote, as b^x can never be zero.

There is no vertical asymptote, although it does get very steep.



Method

We write the graph in the general form: $\boldsymbol{y = b^{x-a} + c}$

where b is the base, how much the graph multiplies by for every one across;
a is the horizontal distance the standard form is shifted. Note the negative.
b is the vertical distance the standard form is shifted.

1) The standard form is $y = b^{x-a} + c$
2) Find the point where the graph is one above its horizontal asymptote.

   This point is (0, 1) for $y = b^x$ before shifts, so one below it is the original (0, 0).
3) The base value, b, is the height of the graph at $x = 1$ before x and y shifts.
4) Put a and c into your equation based on the x and y shifts.
5) Check on calculator. Be sure to bracket in the form $y = b$^$(x - a) + c$

Example

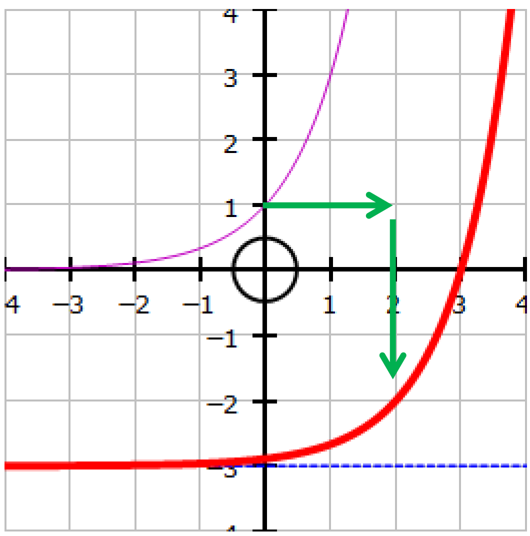
The asymptote is at $y = -3$, shown dotted.

The point (0, 1) therefore has been shifted two to the right and three down from the original (shown with the faint line). So with shifts we have:

$y = b^{x-2} - 3$

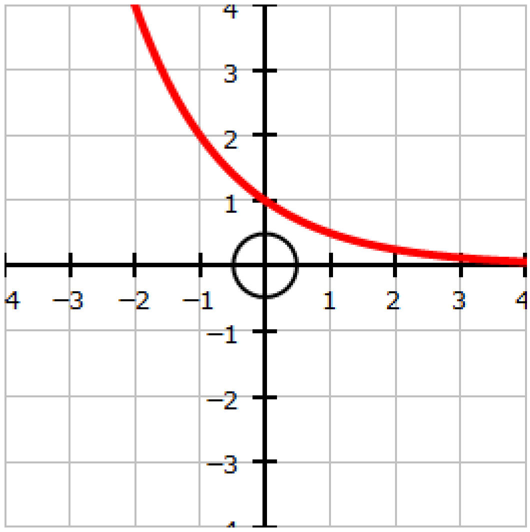
The base is three, as from $x = 2$ to $x = 3$ it goes from one above the asymptote to three above:

$y = 3^{x-2} - 3$



Variations

If the base is less than one, then the curve decreases to the right, often called "decay" curves as opposed to "growth" curves.



For this graph, for each one to the right the y value halves, so the base must be one half.

(0, 1) is not moved so there are no shifts.

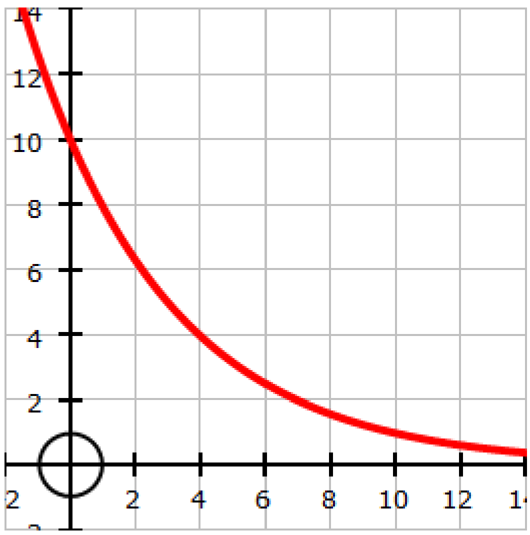
The curve is therefore: $y = 0.5^x$

Because $1/b = b^{-1}$, we can also write such curves with negative powers, giving the same effect.

In this case the curve is also: $y = 2^{-x}$

This is to be avoided, in general, as it also reverses the sign of any x shift.

Excellence: Exponential curves can also be fitted by an appropriate multiplier. This can be useful for some real life situations where the starting value has a meaning.



The general form is y = start × $0.5^{\,t\,\div\,\text{half-life}}$

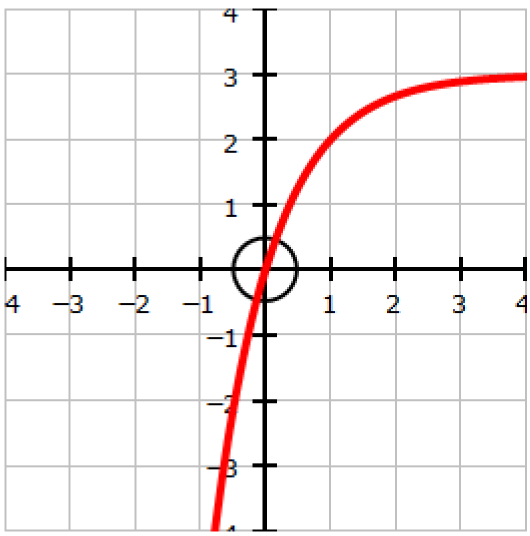
where start is the value for $t = 0$
t is the time in appropriate units
half-life is how long for the value to halve

The graph here starts at 10 on the y-axis and halves every three units across, so is

$$y = 10 \times 0.5^{x/3}$$

Curves like this are common for situations where there is asymptotic decay towards zero but never reaching it (radioactive decay, depreciation).

Excellence: The whole graph can be negative, flipping it in the x-axis, by putting a minus in front. (The base value b itself cannot be negative, as that would involve the square root of a negative.)



The graph here is

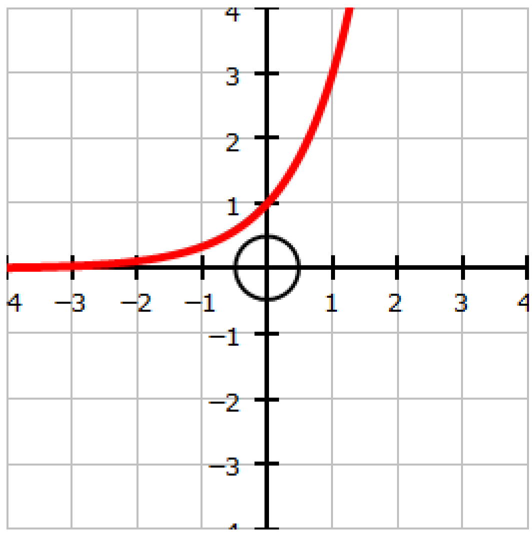
$$y = -\left(3^{-x+1} \right) + 3$$

Growth curves like this, flipped upside down and with a raised asymptote are used to model values that grow towards an upper limit at the asymptote (such as populations that are limited by a set amount of food).

Practice Page 11 – Exponential Graphs

Write equations for the following graphs. Use your graphics calculator to check them.

61



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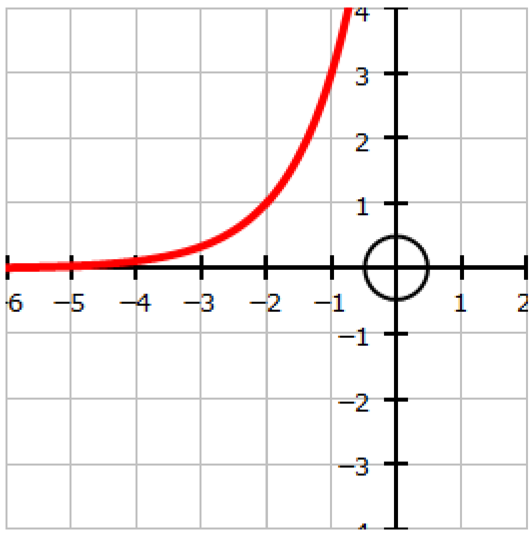

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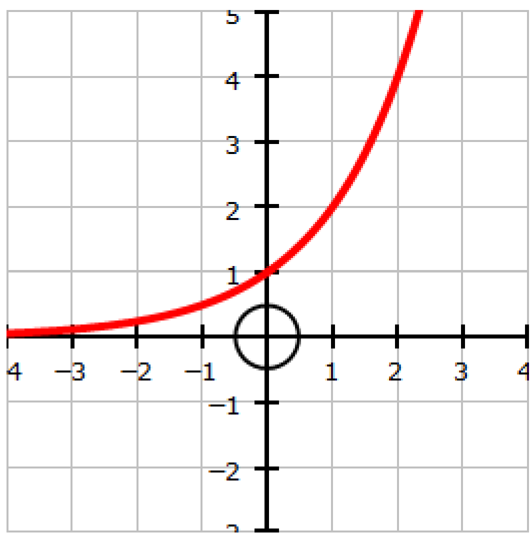

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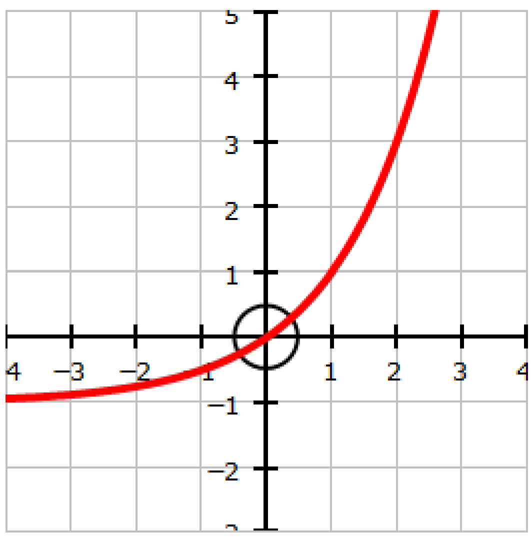

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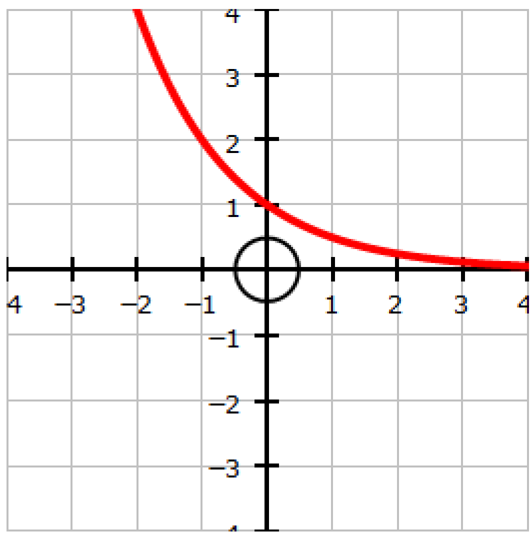

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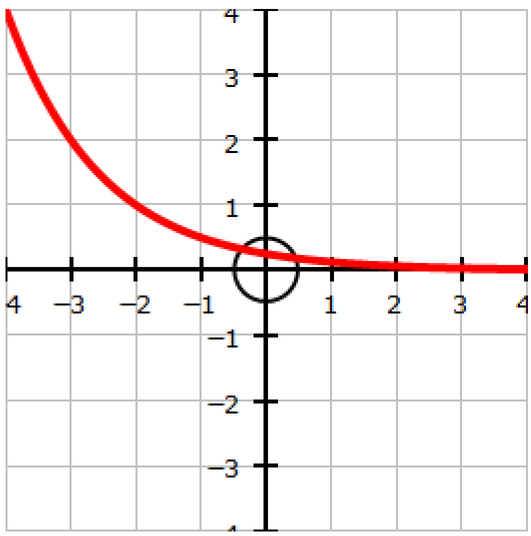

Practice Page 12 – Exponential Graphs

Some harder equations.

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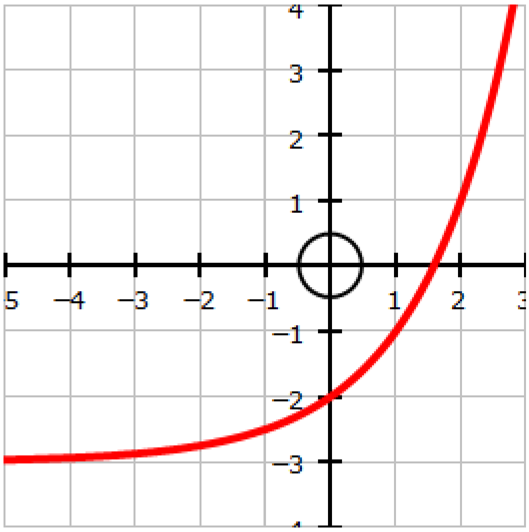

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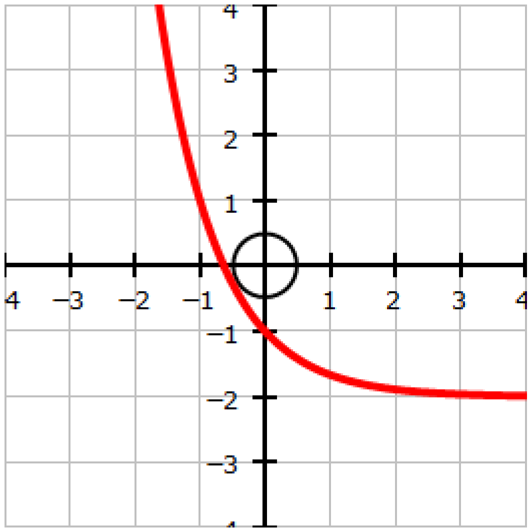

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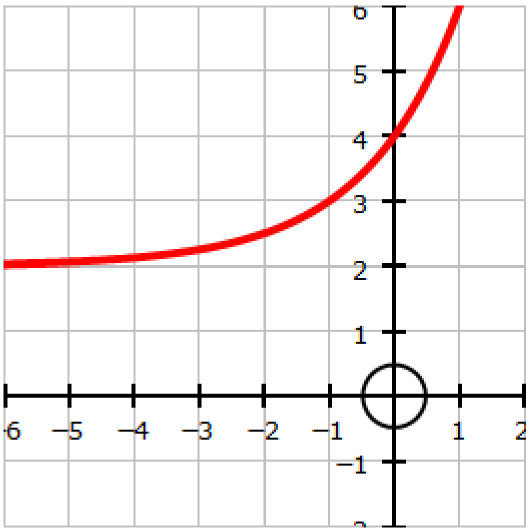

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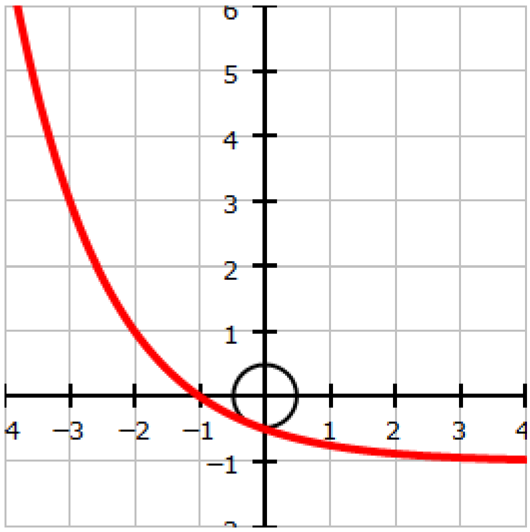

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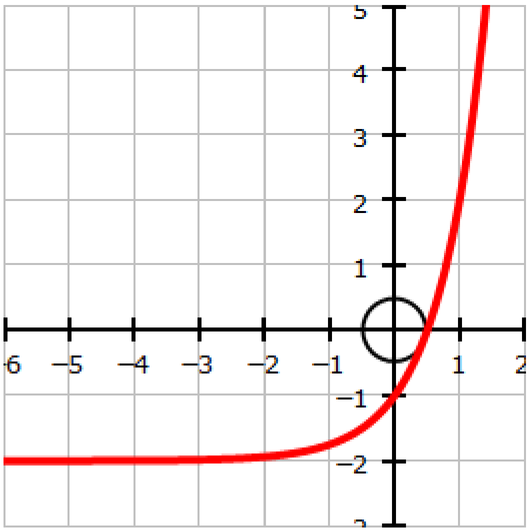

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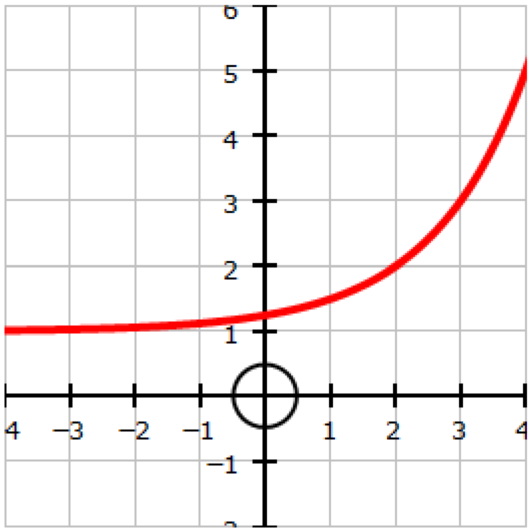


Log Graphs

Log graphs are even more awkward that exponential curves, unless your knowledge of logs is very strong. Mostly just recognising their general features is sufficient at Level 2.

The graph shown is

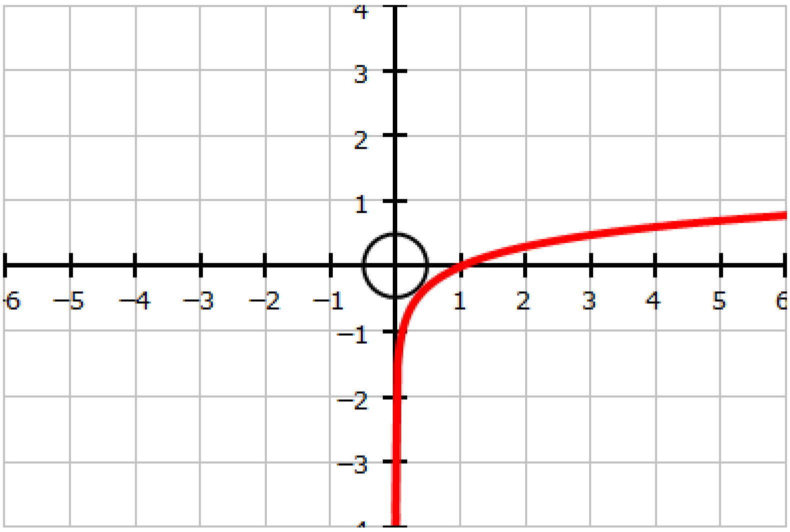
$y = \log x$

The graph is $y = 10^x$ reflected in the line $y = x$.

Because $\log_b 1 = 0$ for all values of b, the point is (1, 0) is always the y-intercept for $y = \log_b(x)$

There is a vertical asymptote, as log 0 is undefined.

There is no horizontal asymptote, although it does get very flat.



Method

We write the graph in the general form: $\boldsymbol{y = m \log_b(x - a) + c}$

where b is the base of the log (which is 10 on the calculator);
m is a multiplier.
a is the horizontal distance the standard form is shifted. Note the negative.
c is the vertical distance the standard form is shifted.

1) The standard form is $y = m \log (x - a) + c$
2) Find the point where the graph is one to the right of the horizontal asymptote.
   This point is (1, 0) for $y = \log x$ before shifts, so one left is the original (0, 0)
3) Put in a known value to get the m value.
5) Check on calculator. Be sure to bracket in the form $y = \log(x - a) \div \log(\text{b}) + c$

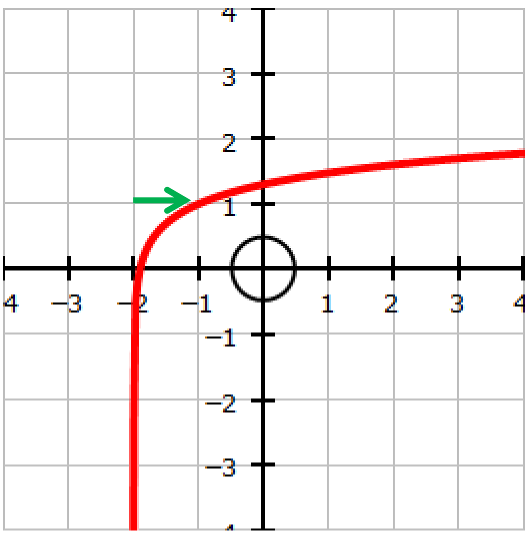
Example

The asymptote is at $x = -2$ and the graph has been shifted one up, the arrow showing where the graph is one away from the asymptote.

$y = m \log (x + 2) + 1$

There is no m multiplier in this case (although it is hard to tell at this scale):

$y = \log (x + 2) + 1$



Variations

The steepness of the graph can be changed by changing the base of the log, in the same manner as working out the base of an exponential graph, rather than changing the m value.

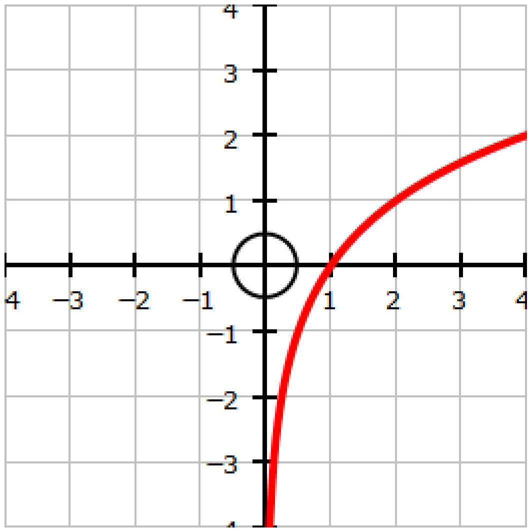
The vertical asymptote is the y-axis and the graph crosses at (1, 0) so there are no x or y shifts.

The curve is therefore:

$$y = m \log (x)$$

We see that for each increase in y the distance from the asymptote doubles, so the base is 2:

$$y = \log_2(x)$$



We can't write $\log_2 x$ on our calculators though. We can mimic that by using the relationship $\log_b x = \log_{10} x / \log_{10} b$, so we type in

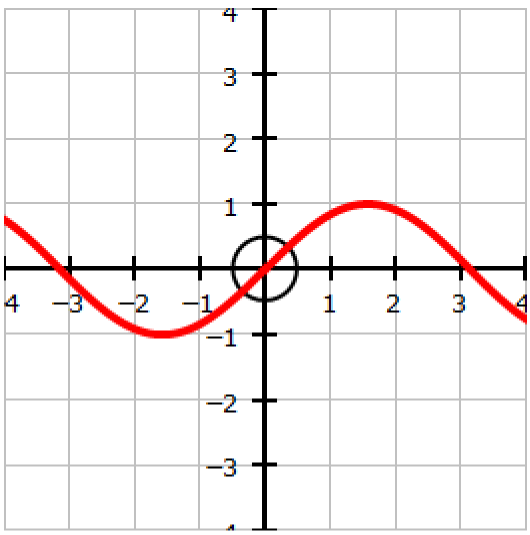
$$y = \log(x) \div \log(2)$$

There aren't a lot of occasions where log graphs appear in nature or economic life. When they do it is usually a situation where we think of them in terms of each doubling of the x value is only half as effective as the previous one (for example, increasing concentration of dye related to the amount of light let through).

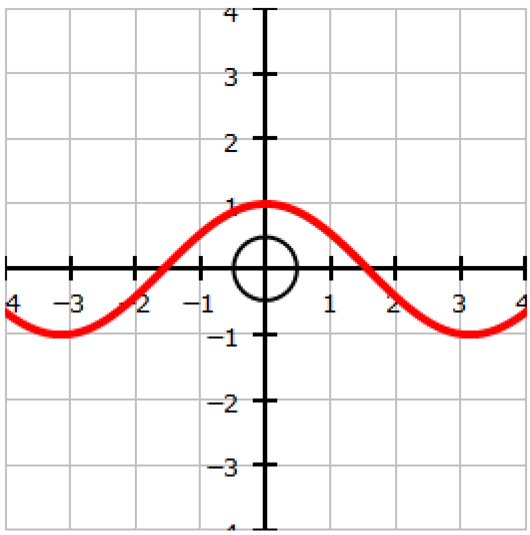
Practice with Log curves is left only as an extension exercise.

Sine and Cosine Curves

The trigonometric curves for Sine and Cosine have a lot of important uses, modelling a large number of natural phenomena. They are technically part of the Level 2 course, but are covered in great detail in Level 3 in the Calculus course, so won't be covered here.



$y = \sin (x)$



$y = \cos (x)$

The rules for how these graphs are shifted and altered are exactly the same as for the other curves in this unit.

Functions

A function is a mathematical process which gives at most one output value for any input.

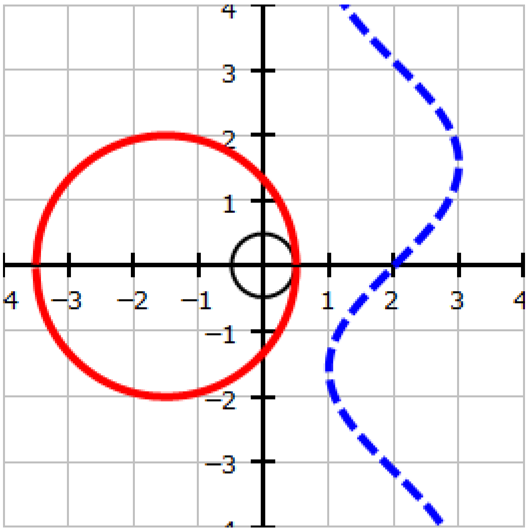
Graphically a function can be determined by seeing if there are any vertical lines which pass through two y values – if so, it is not a function.

Examples that are **not** functions:

$x = -1$ gives two y results for the circle

$x = 2$ give three y values for the dotted curve

Note that a function does not always return a value for every x. There is no y value for x = –2 for y = log(x), for example.



It is usually easy to tell a function because they can be written in the form y = something. For example, $y = x^2$ or $y = \log(x)$.

A circle however is written $x^2 + y^2$ = constant, and the y^2 gives the game away that it is not a function. (We can write a circle as $y = \pm\sqrt{c - x^2}$ but the ± gives us a sure sign that we are getting two outputs, not one, so it is not a function.)

At Level 2 we stick to functions– that is for every x value there is at most one y value.

Function Notation – $f(x)$

We can write any function using function notation where $y = f(x)$. We graph it so that the $f(x)$ axis replaces the y axis, giving exactly the same result as a x/y graph.

Writing $f(x) = 2x^2 + 7x$ is exactly the same as writing $y = 2x^2 + 7x$. They are both functions, even if only one uses the function notation.

Different functions can be written with different letters, e.g. $g(x)$ and $h(x)$, but $f(x)$ is usual.

You are used to using this style notation for functions, even if you haven't recognised it as such in the past. We write $y = \log(x)$ and $y = \sin(x)$ where log and sine are both functions.

Function notation is heavily used in Level 2 Calculus, so is important to learn.

Polynomials

A **polynomial** is anything that can be written in the form $y = ax^m + bx^n \ldots + gx + h$, where the powers are all positive integers. All polynomials are functions.

For example any cubic can be expanded out and written in the form $y = ax^3 + bx^2 + cx + d$ and any parabola can be written in the form $y = ax^2 + bx + c$.

Technically lines are polynomials, but we generally don't include them in that term.

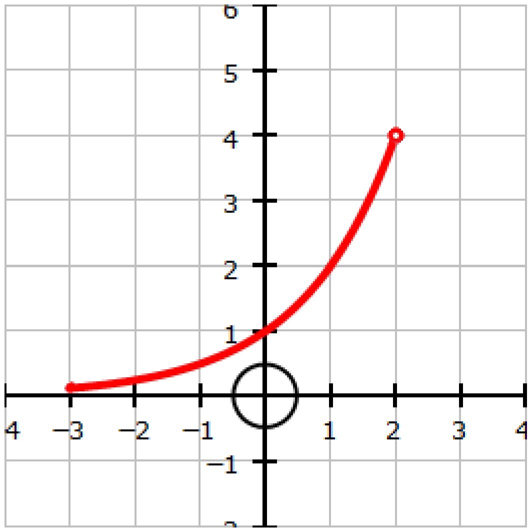
Domain and Range

The **domain** of a function is those x-values for which there is also a y value. The domain of $f(x) = x^2$ is all values of x, because there is no value of x for which x^2 has no value.

However the domain of $f(x) = \log(x)$ is $x > 0$, because $\log(x)$ has no meaning for $x \leq 0$.

We can also give any function a specified domain, so that it is clipped outside those values.

Graphed is $f(x) = 2x$ for $-3 \leq x < 2$



The convention is that points that are included in the function, such as (−3, 0.125), are shown solid, whereas points that the graph goes up to but that are not included themselves, such as (2, 4) in this case, are shown hollow.

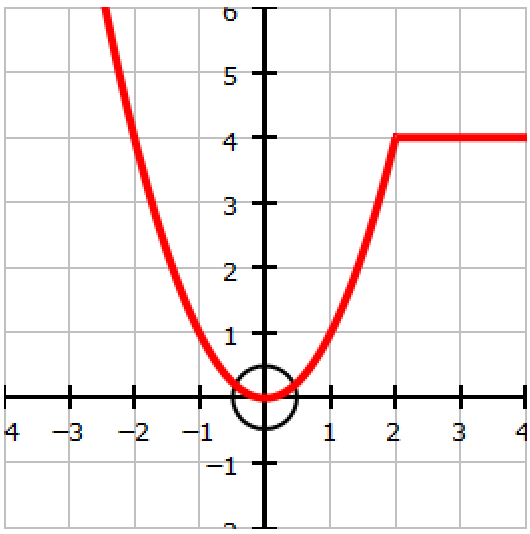
The **range** of a function is the y values that it can take. The range for the function shown to the right is $0.125 \leq y < 4$

We don't use range much, and especially not to limit functions, as it can be ambiguous.

Piecewise functions

We can also use our control of domain to "glue together" two different functions, so that they combine into a new function. Such functions are defined in their difference pieces, and so are called "piecewise".

The curve to the left is:



$$
\begin{aligned}
f(x) &= x^2 && \text{for } x < 2 \\
&= 4 && \text{for } x \geq 2
\end{aligned}
$$

Be particularly careful to define the point where the two different pieces join. One side will have an equals, the other won't, to avoid the point being defined twice.

Piecewise functions do not need to connect. If they don't it is even more important to define carefully the area around the change.

Practice Page 13 – Domain and Range

Write the domains for the following functions. Assume any that go off page are infinite in that direction. (If you can, write the equation for the function itself as well as the domain.)

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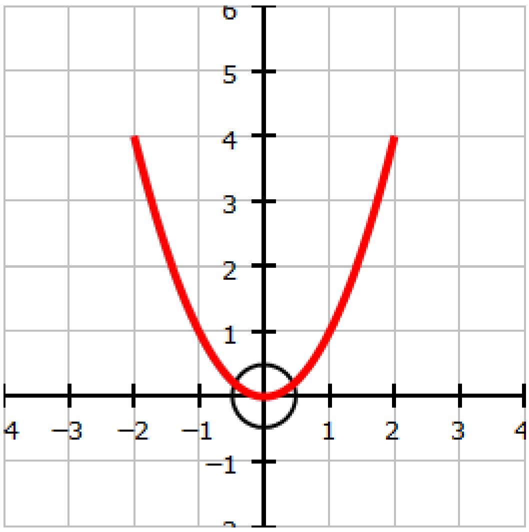

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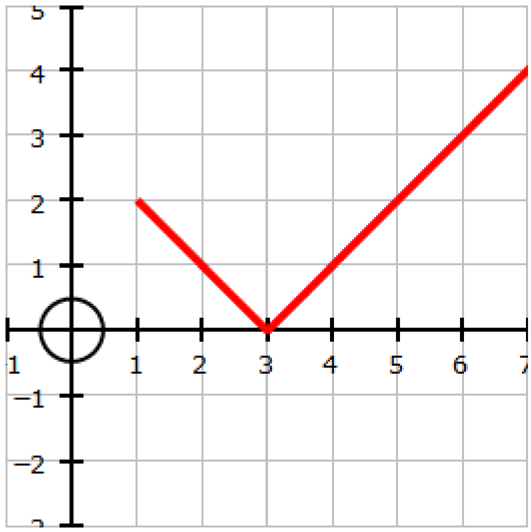

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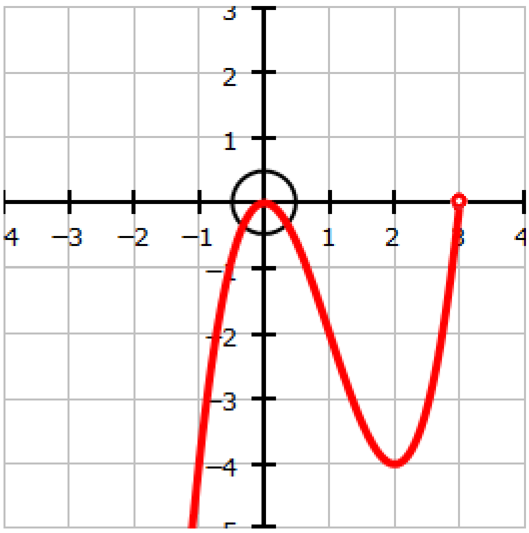

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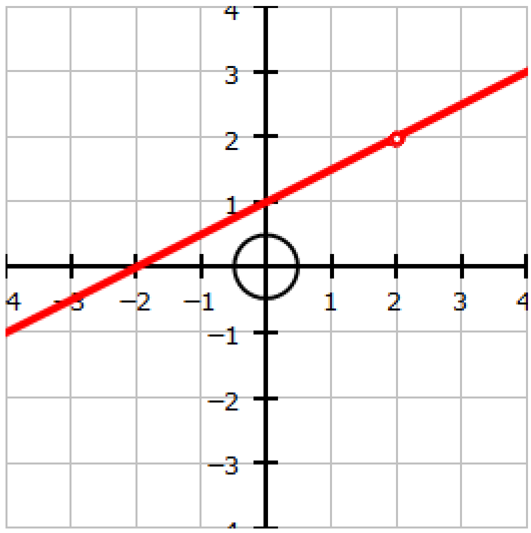

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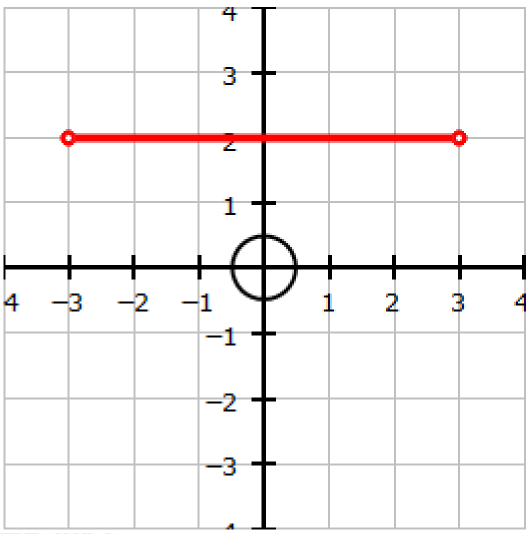

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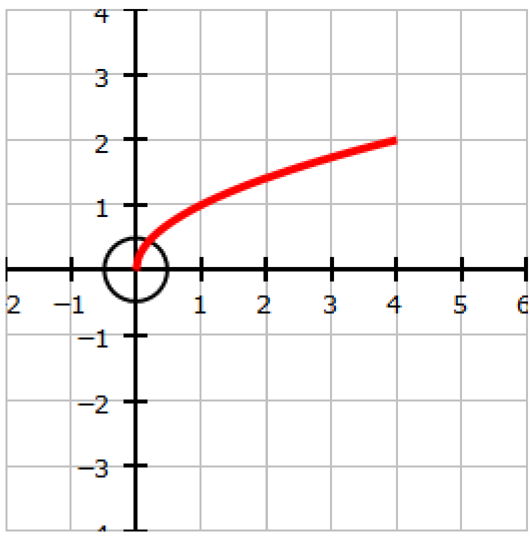

Practice Page 14 – Piecewise

Write the domains for the following functions, with appropriate domain restrictions. They are made up of simple parabolas and lines.

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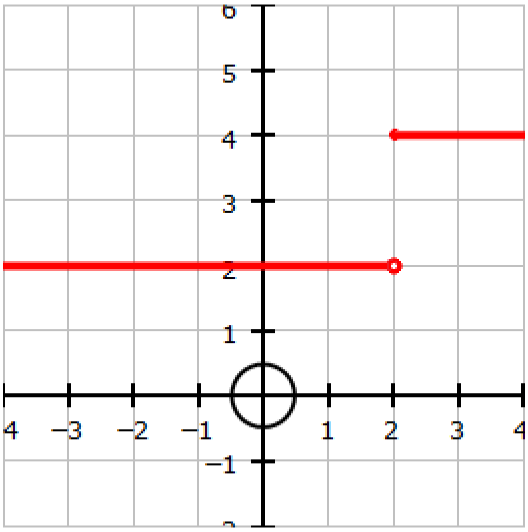

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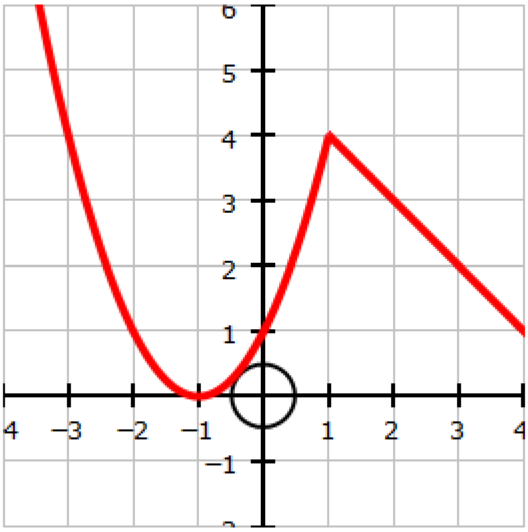

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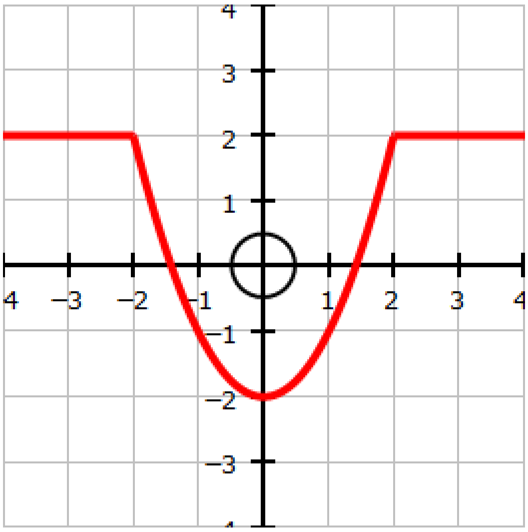

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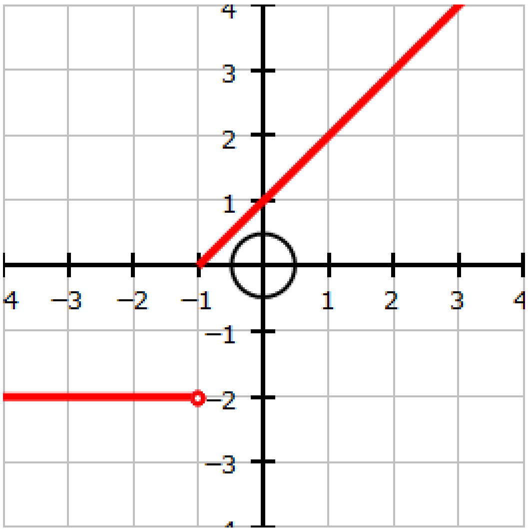

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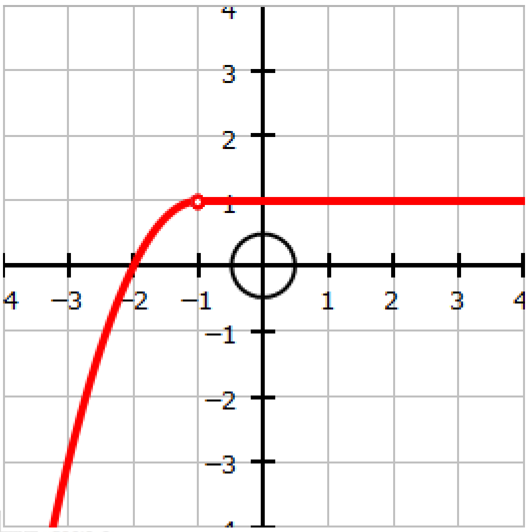

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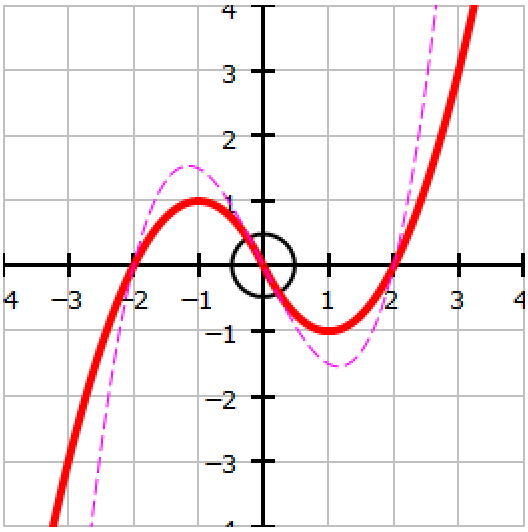
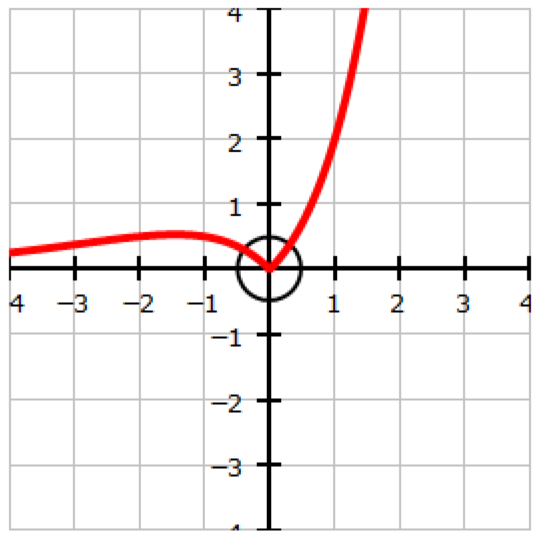


Describing Changes using Function Notation

Here is a function, $f(x)$.

We know that it is a function because for every x value there is one, and only one y value. (It isn't smooth, since there is a sharp point at (0, 0) but functions don't have to be.)

It is, as it happens, $f(x) = | x\ 2^x |$

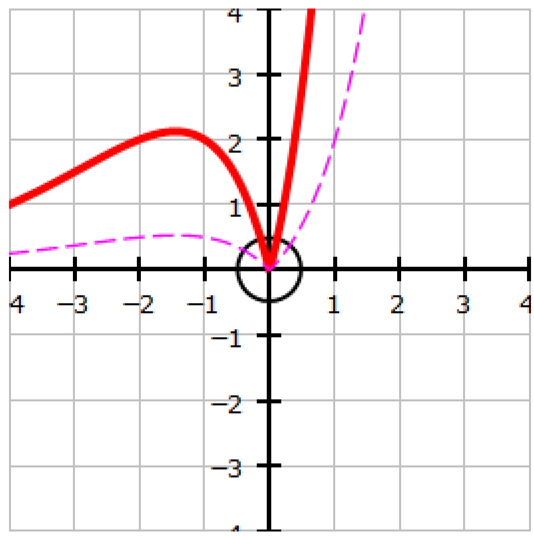


Putting a multiplier completely **outside** the whole function always acts to multiply the distances from the x-axis by that amount.

So if we take $f(x)$ and change it to $4\ f(x)$ all the y values will be 4× as big.

In this case $f(x) = 4\ (\ | x\ 2^x |\ )$ as pictured to the right has a turning point four times as high, and the lines are all four times as steep, compared to the original.

Putting a multiplier ***inside*** *the function has very different effects depending on the function, but will only rarely be a simple multiplication.*

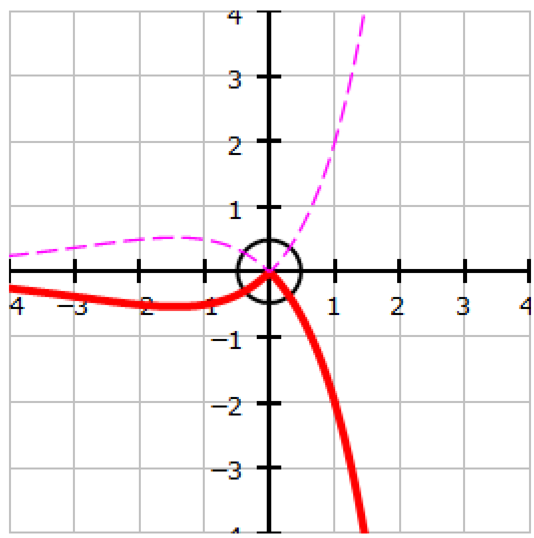


If the multiplier of the function has a negative sign, then the graph is reflected in the x-axis.

So if we take $f(x)$ and change it to $-\ f(x)$ all the y values will made into their negatives.

So $f(x) = -\ | x\ 2^x |$ as pictured to the right has been flipped compared to the original.

Excellence*: replacing x with $-x$ in every position* ***inside*** *the function will reflect in the y–axis, rather than x-axis, so $y = | -x\ 2^{-x} |$ will reflect in the y-axis. There are usually easier ways to achieve that.*



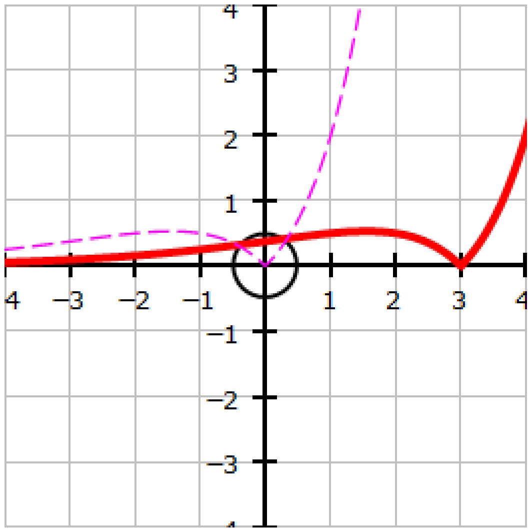
If every instance of the x value being replaced by $(x - a)$ **inside** the function, where a is any numerical value, will shift the graph by a distance of a to the right *without any other changes*.

So if we take $f(x)$ and change it to $f(x-a)$ we get an x-shift.

In this case $f(x) = |\ (x-3)\ 2^{x-3}\ |$ as pictured to the right has been moved three units right.

*This always works, but it must be **every** case of x being replaced and generally requires brackets if not there already.*

So we can shift $y = x\ (x - 2)\ (x - 4)$ one place to the right by making it $y = (x - 1)\ (x - 3)\ (x - 5)$



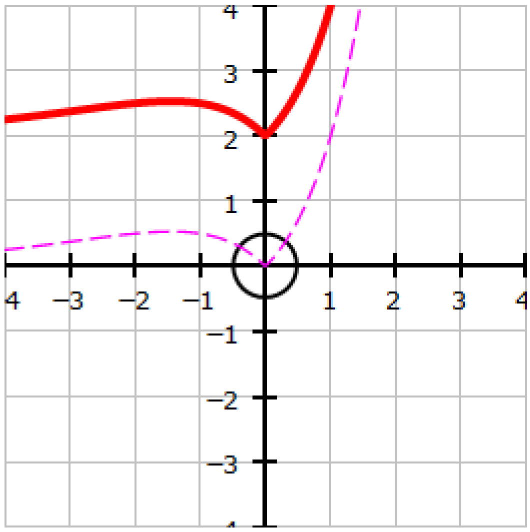
Adding a constant **after** the function will raise all values by that constant. (The shift to the y is actually negative, but as we write it on the other side of the equation it becomes positive.)

So we take $y = f(x)$ and change it to $y - c = f(x)$ to raise the graph.

But we write that as $y = f(x) + c$.

All the y values will now be c higher up.

In this case $f(x) = |\ x\ 2^x\ | + 2$ as pictured to the right has every value two higher, but the shape unchanged.



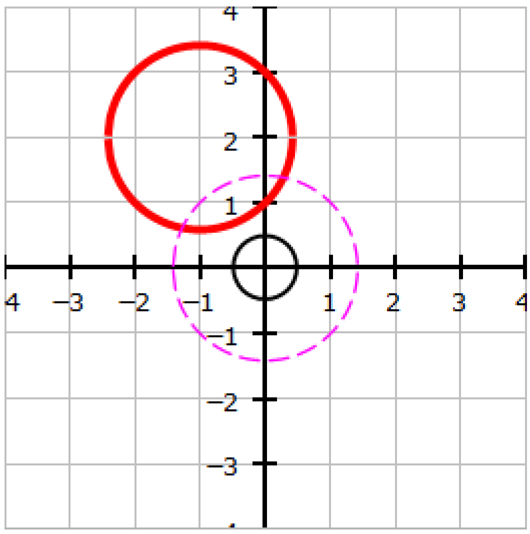
This application of shifts to x and y values as negatives works for **all** graphs, not just functions.

The original, dotted, circle is $x^2 + y^2 = 2$.

It has been moved one left, so $x \to x - (-1)$ and two higher, so $y \to y - 2$.

The solid circle is $(x + 1)^2 + (y - 2)^2 = 2$

Note though, because non-functions don't work on the basis of y = something, they cannot be made larger or reflected as simply as functions can.



Answers

Fractions can replace decimals. Brackets can be in any order.

Practice Page 1 – Parabolas

1 $y = (x - 1)(x - 3)$ or $y = (x - 2)^2 - 1$

2 $y = 2(x + 1)^2 - 3$

3 $y = 0.25(x + 2)^2$

4 $y = -2x(x - 2)$ or $y = -2(x - 1)^2 + 2$

5 $y = 2(x + 1)^2 + 1$

6 $y = (x - 3)^2 + 1$

Practice Page 2 – Parabolas

7 $y = -3(x - 2)^2 + 4$

8 $y = -2(x - 5)^2$

9 $y = 1.5(x + 1)^2 - 1$

10 $y = 0.25(x + 2)^2 + 1$

11 $y = 0.5(x - 4)^2 + 3$

12 $y = \frac{2}{3}(x - 1)^2 - 3$

Practice Page 3 – Square Root Graphs

13 $y = \sqrt{x+2} - 1$

14 $y = \sqrt{x} + 2$

15 $y = -3\sqrt{x} + 3$

16 $y = \sqrt{x+1} - 3$

17 $y = 2\sqrt{x-1}$

18 $y = -\sqrt{x+3}$

Practice Page 4 – Square Root Graphs

19 $y = 1.5\sqrt{x+4}$

20 $y = 1.5\sqrt{x} - 3$

21 $y = -2\sqrt{x+1} + 3$

22 $y = 0.5\sqrt{x+4} + 1$

23 $y = -2\sqrt{x-1} + 2$

24 $y = 3\sqrt{-x+2} - 2$

Practice Page 5 – Cubic Graphs

25 $y = (x + 1)(x - 1)(x - 3)$

26 $y = 0.5\,(x + 2)(x - 1)(x - 3)$

27 $y = -\,(x + 3)(x + 2)\,x$

28 $y = -\,(x + 2)(x - 1)(x - 2)$

29 $y = -0.25\,(x + 2)(x - 2)(x - 4)$

30 $y = (x + 4)(x + 2)^2$

Practice Page 6 – Cubic Graphs

31 $y = \frac{1}{6}(x + 2)(x - 3)^2$

32 $y = \frac{3}{2}(x + 2)^2(x + 1)$

33 $y = \frac{-3}{2}(x + 3)(x + 2)\,x$

34 $y = \frac{2}{3}(x + 3)(x + 2)(x - 1)$

35 $y = -4\,(x + 1)(x - 1)^2$

36 $y = (x - 1)^3$

Practice Page 7 – Hyperbolas

37 $y = \frac{1}{x+3}$

38 $y = \frac{1}{x} - 2$

39 $y = \frac{3}{x}$

40 $y = \frac{1}{x+2} - 1$

41 $y = \frac{-2}{x} + 1$

42 $y = \frac{2}{x-2}$

Practice Page 8 – Hyperbolas

43 $y = \frac{2}{x-2} + 2$

44 $y = \frac{-1}{x+3} + 2$

45 $y = \frac{-3}{x} - 3$

46 $y = \frac{2}{x-1} + 1$

47 $y = \frac{8}{x} + 2$

48 $y = \frac{-4}{x+2} - 4$

Answers Contd

Practice Page 9 – Absolute Value

49 $y = | x - 2 | - 1$

50 $y = | x + 1 | + 2$

51 $y = -2 | x |$

52 $y = 4 | x | - 2$

53 $y = | x - 1 | + 3$

54 $y = - | x + 2 |$

Practice Page 10 – Absolute Value

55 $y = 2 | x + 2 | - 1$

56 $y = - 3 | x + 1 | + 2$

57 $y = 0.5 | x + 2 | - 1$

58 $y = \frac{-1}{3} | x + 1 | + 2$

59 $y = 5 | x - 1 | - 3$

60 $y = 1.5 | x - 2 | + 1$

Practice Page 11 – Exponential Graphs

61 $y = 3^{x + 1} + 3$

62 $y = 3^{x + 2}$

63 $y = 2^x$

64 $y = 2^x - 1$

65 $y = 0.5^x$

66 $y = 0.5^{x + 2}$

Practice Page 12 – Exponential Graphs

67 $y = 2^x - 3$

68 $y = (\frac{1}{3})^x - 2$

69 $y = 2^{x + 1} + 2$

70 $y = 0.5^{x + 1} - 1$

71 $y = 4^x - 2$

72 $y = 2^{x - 2} + 1$

Practice Page 13 – Domain and Range

73 $y = x^2$ for $-2 \le x \le 2$

74 $y = | x - 3 |$ for $1 \le x$

75 $y = x^2 (x - 3)$ for $x < 3$

76 $y = 0.5x + 1$ for $x \ne 2$

77 $y = 2$ for $-3 < x < 3$

78 $y = \sqrt{x}$ for $0 \le x \le 4$

Practice Page 14 – Piecewise

For questions 80, 81 and 84 the equals can go on either side of the joins.

79 $y = 2$ for $x < 2$
$= 4$ for $x \ge 2$

80 $y = (x + 1)^2$ for $x < 1$
$= -x + 5$ for $x \ge 1$

81 $y = 2$ for $x < -2$
$= x^2 - 2$ for $-2 \le x \le 2$
$= 2$ for $x > 2$

82 $y = -2$ for $x < -1$
$= x + 1$ for $x \ge 1$

83 $y = -(x + 1)^2 + 1$ for $x < -1$
$= 1$ for $x > -1$

84 $y = -(x + 1)^2 + 1$ for $x < 0$
$= (x - 1)^2 - 1$ for $x \ge 0$
Note the different shape of the cubic $y = 0.5 (x + 2) x (x - 2)$