Level 2 Solving Simultaneous Equations

Students need to be able to solve both simple linear and quadratic equations with accuracy **before** they attempt to solve simultaneous equations.

Students also need to be able to expand brackets, including quadratics, with accuracy.

Terminology

Solving a pair of **simultaneous** equations means finding the values for the variables so that both equations are true. Graphically this the points where they intersect.

A **variable** is one of the unknown values in your equations. Generally x and y, but they can be any letter, especially when representing real world values.

A **linear** equation is one written terms of x and y to the power of one only. There are three ways these can be written, all identical in effect, and all of which form lines when plotted on a Cartesian grid (which is why they are "linear" equations).

$y = 3x + 4$ $\qquad$ $5x + 6y = 30$ $\qquad$ $4x - 3y + 6 = 0$

The Substitution Method:

We cannot solve an equation with two variables, so we must somehow remove a variable. We do this by "substituting" one equation into the other.

The technique is the same regardless of what form the equations are in.

1) One equation is re-written so that the first variable is given in terms of the second.
2) This equation is used to replace the first variable in the other equation.
3) The new equation is solved for the one remaining variable left in it. There may be more than one solution.
4) Returning to the original equations the first variable is found.
5) The answer is given in terms of both variables, as a pair or pairs.

The alternative "elimination" method is not suitable in general for non-linear equations.

Systems of Linear Equations

Students should not move on to more difficult questions until they understand the techniques properly, which means doing linear problems until they become routine.

When both equations are given in the form $y =$ then solving the equation becomes a matter of making the other sides of the equations equal.

Solve where ① $y = 5 - x$ meets ② $y = 4x + 2$

step 1)	$y = 4x + 2$	already done
step 2)	$5 - x = 4x + 2$	y ① $= y$ ② so we have only xs in our equation now
step 3)	$5 - 2 = 4x + x$	expanding the brackets
	$3 = 5x$	rearranging
	$x = 0.6$	solved
step 4)	$y = 5 - 0.6 = 4.4$	finding the other variable
	lines meet at (0.6, 4.4)	

Otherwise the value of one is replaced inside the other.

It is vital that the term substituting for a variable must be put in brackets because it replaces the whole of that variable.

Solve where ① $y = 2x + 5$ meets ② $2x + 3y = 10$

step 1)	$y = 2x + 5$	is already done for us
step 2)	$2x + 3(x + 5) = 10$	① is used to replace the y in ②. Note the brackets
step 3)	$2x + 3x + 15 = 10$	expanding the brackets
	$2x + 3x = 10 - 15$	rearranging
	$x = {}^-1$	solved
step 4)	$y = 2 \times {}^-1 + 5 = 3$	finding the other variable
	lines meet at (${}^-1$, 3)	

Students need to recall the basic theory of lines:

a) Parallel lines have the same gradient, which means they have the same coefficients of x and y

e.g. lines parallel to $y = 3x + 9$ have the general form $y = 3x + k$

and

e.g. lines parallel to $2x + 5y = 10$ have the general form $2x + 5y = k$

b) Perpendicular lines have gradients that multiplied together equal ${}^-1$.

e.g. $y = 0.5x + 3$ and $y = {}^-2x - 5$ are perpendicular as $0.5 \times {}^-2 = {}^-1$

Systems of a Parabola and a Line

Will give a quadratic during the working.

It is almost always easier to substitute out the *y* values (because substituting the *x* values involves squaring the new terms, which makes the working much more difficult).

Solve where $y = (x + 5)^2$ meets $y = x + 7$

step 1)	$y = x + 7$	already done
step 2)	$(x + 5)^2 = x + 7$	$y = y$ when the equations intersect
step 3)	$x^2 + 10x + 25 = x + 7$	expanding the brackets
	$x^2 + 10x + 25 – x – 7 = 0$	quadratic, so all terms to one side
	$x^2 + 9x + 18 = 0$	simplifying
	$x = {}^{-}6$ or ${}^{-}3$	solved by factorisation (or on graphics)
step 4)	$y = {}^{-}6 + 7 = 1$	the *y* value for the first solution
	$y = {}^{-}3 + 7 = 4$	the *y* value for the second solution
	equations meet at $({}^{-}6, 1)$ and at $({}^{-}3, 4)$	

If fractional terms are introduced, multiply them out as early as possible, remembering to multiply <u>every</u> term by the same amount

Solve where $y = ½(x – 4)^2 – 2$ meets $x + 2y = 6$

step 1)	$y = 3 – ½x$	
step 2)	$3 – ½x = ½(x – 4)^2 – 2$	$y = y$ when the equations are simultaneous
step 3)	$6 – x = (x^2 – 8x + 16) – 4$	multiply everything by 2 to get rid of the ½ s
	$0 = x^2 – 7x + 6$	simplified
	$x = 1$ or 6	solved by factorisation (or on graphics)
step 4)	$y = 6 – 2 × 0 = 6$	the *x* value for the first solution
	$y = 6 – 2 × 2.5 = 1$	the *x* value for the second solution
	equations meet at (6, 0) and at (1, 2.5)	

or

Solve where $y = ½(x – 4)^2 – 2$ meets $x + 2y = 6$

step 1)	$2y = (x – 4)^2 – 4$	multiply everything by 2 to get rid of the ½
	$x = 6 – 2y$	(as opposed to $y = 3 – ½x$)
step 2)	$2y = (6 – 2y – 4)^2 – 2$	*x* in new ① is substituted by *x* in new ②
step 3)	$2y = (2 – 2y)^2 – 4$	multiplying by 2 and expanding the brackets
	$0 = 4y^2 – 10y$	expand and simplify
	$y = 0$ or 2.5	solved by factorisation (or on graphics)
step 4)	$x = 6 – 2 × 0 = 6$	the *x* value for the first solution
	$x = 6 – 2 × 2.5 = 1$	the *x* value for the second solution
	equations meet at (6, 0) and at (1, 2.5)	

Systems of a Hyperbola and a Line

Will give a quadratic during the working.

If the hyperbola is given in the form xy = c then either the x or y can be substituted out directly.

Solve where $xy = 8$ meets $y = 4x - 4$

step 1)	$y = 4x - 4$	is already done for us
step 2)	$x\,(4x - 4) = 8$	$y = y$ when the lines intersect
step 3)	$4x^2 - 4x = 8$	expanding the brackets
	$4x^2 - 4x - 8 = 0$	quadratic, so all terms to one side
	$x^2 - x + 2 = 0$	simplifying
	$x = {}^{-}1$ or 2	solved by factorisation (or on graphics)
step 4)	$y = 4 \times {}^{-}1 - 4 = {}^{-}8$	the y value for the first solution
	$y = 4 \times 2 - 4 = 4$	the y value for the second solution
	equations meet at $({}^{-}1, {}^{-}8)$ and at $(2, 4)$	

For a hyperbola given in a fractional form multiplying across the denominator is the key first step. Remember that the denominator of a fraction <u>always</u> acts as if it is in brackets.

Solve where $y = \dfrac{-1}{x-2}$ meets $y = 2 - x$

step 1)	$y\,(x - 2) = {}^{-}1$	① rearranged – note brackets
step 2)	$(2 - x)(x - 2) = {}^{-}1$	y in new ① is substituted by y in ②
step 3)	$2x - 4 - x^2 + 2x = {}^{-}1$	expanded
	$0 = x^2 - 4x + 3$	quadratic, so all terms to one side
	$x = 1$ or 3	solved by factorisation (or on graphics)
step 4)	$y = 2 - 1 = 1$	the x value for the first solution
	$y = 2 - 3 = {}^{-}1$	the x value for the second solution
	equations meet at $(1, 1)$ and at $(3, {}^{-}1)$	

In general the working is not more difficult if x is substituted out, and sometimes that is easier.

Solve where $y = \dfrac{4}{x+1}$ meets $x - 4y + 1 = 0$

step 1)	$y\,(x + 1) = 4$	① rearranged – note brackets
	$x = 4y - 1$	② rearranged, less awkwardly than $y = ¼x + ¼$
step 2)	$y\,(4y - 1 + 1) = 4$	x in new ① is substituted by x in new ②
step 3)	$4y^2 - 4 = 0$	quadratic, so all terms to one side
	$y^2 - 1 = 0$	
	$y = {}^{-}1$ or 1	solved by factorisation (or on graphics)
step 4)	$x = 4 \times {}^{-}1 - 1 = {}^{-}5$ and $x = 4 \times 1 - 1 = 3$	
	equations meet at $({}^{-}1, {}^{-}5)$ and at $(1, 3)$	

Systems of a Circle and a Line

Will give a quadratic during the working.

The simplest circle is of the form $x^2 + y^2 = r^2$ where the origin is the centre of circle radius = r

The more general form is $(x - a)^2 + (y - b)^2 = r^2$ where the centre is (a, b).

In general try to substitute out the y value if that is possible. Often it is already given directly in the linear equation, making that the obvious method.

Solve where $(x + 1)^2 + (y - 2)^2 = 16$ meets $y = -x + 5$

step 1)	$y = -x + 5$	is already done for us
step 2)	$(x + 1)^2 + (-x + 5 - 2)^2 = 16$	y in ① is substituted by y in ②
	$(x + 1)^2 + (3 - x)^2 = 16$	simplified
step 3)	$x^2 + 2x + 1 + 9 - 6x + x^2 = 16$	
	$2x^2 - 4x - 6 = 0$	quadratic, so all terms to one side
	$x^2 - 2x - 3 = 0$	simplifying
	$x = {}^{-}1$ or 3	solved by factorisation (or on graphics)
step 4)	$y = -{}^{-}1 + 5 = 6$	the y value for the first solution
	$y = -3 + 5 = 2$	the y value for the second solution
	equations meet at $({}^{-}1, 6)$ and at $(3, 2)$	

Extreme care must be taken when expanding brackets.

Substitute in the new value, then simplify inside the bracket first. Only then expand.

$(y + 1)^2$ if $y = 2x - 5$

$= ([2x - 5] + 1)^2$

$= (2x - 5 + 1)^2$

$= (2x - 4)^2$

$= 4x^2 - 8x - 8x + 16$

$= 4x^2 - 16x + 16$

Circles are awkward to draw on graphics calculators so checking is harder. This makes it all the more important to square terms carefully.

Circles can also be written in the form: $ax^2 + ay^2 + bx + cy = 0$ which means that the substitution of x or y has to occur twice.

Hyperbolas with diagonal asymptotes can also be written in the general form $(x - a)^2 - (y - b)^2 = c$. It is important to be careful to make each part of the $-y^2$ term negative, but in all other respects they solve like circles.

When to Substitute *x* and When to Substitute *y*

The answers will be the same either way, but most students find it more comfortable to replace the y values (so that only the x values remain) since they are used to solving for x.

Sometimes it is easier to do it the other way, generally to avoid awkward fractions.

Merit students must be comfortable with substituting out either variable.

If the only term without a coefficient is one of the x values, then sometimes that makes using that the easiest.

Solve where $x + 4y = 5$ meets $3x + 2y = 12$

step 1)	$x = 5 - 4y$	① gives x in terms of y without any fractions
step 2)	$3(5 - 4y) + 2y = 12$	new ① is used to replace the x value in ②
step 3)	$15 - 12y + 2y = 12$	expanding the brackets
	${}^{-}10y = {}^{-}3$	rearranging
	$y = 0.3$	solved
step 4)	$x = 5 - 4 \times 0.3 = 3.8$	finding the other variable using the rearranged ①
	lines meet at (3.8, 0.3)	

It would have been quite awkward to write either ① or ② in the form $y =$ and then substitute that into the other, but it was straightforward to rewrite ① to find x.

Generally you want to avoid awkward square terms, so substituting for x in parabolas is almost always wrong.

For hyperbolas neither term is squared, so use the easiest form of the linear equation to work with.

For circles both the x and y terms are squared, so again rearrange the linear equation to give whichever form gives the easier values to work with.

Solve where $(x + 3)^2 + (y - 2)^2 = 25$ meets $x + 3y = 8$

step 1)	$x = 8 - 3y$	much less awkward than $y = {}^{-}\frac{1}{3}x + 2\frac{2}{3}$
step 2)	$(8 - 3y + 3)^2 + (y - 2)^2 = 25$	x in ① is substituted by x in new ②
	$(11 - 3y)^2 + (y - 2)^2 = 25$	simplified
step 3)	$121 - 66y + 9y^2 + y^2 - 4y + 4 = 25$	
	$10y^2 - 70y + 100 = 0$	quadratic, so all terms to one side
	$y^2 - 7y + 10 = 0$	simplifying
	$y = 2$ or 5	solved by factorisation (or on graphics)
step 4)	$x = 8 - 3 \times 2 = 2$	the x value for the first solution
	$x = 8 - 3 \times 5 = {}^{-}7$	the x value for the second solution
	equations meet at (2, 2) and at (${}^{-}7$, 5)	

Be careful that if you find the y values initially that you put them into the solution in the right place.

It is extremely easy to, for example, find $y = 3$ first and then solve for $x = 1$ and write the answer as (3, 1) rather than (1, 3).

Number of Solutions (Merit)

For lines intersecting parabolas, hyperbolas and circles the working yields a quadratic to solve.

The number of solutions depends on the $b^2 - 4ac$ term of the quadratic equation:

$b^2 - 4ac > 0$ means two (real) solutions

$b^2 - 4ac = 0$ means one solution

$b^2 - 4ac < 0$ means none.

This can be used to solve situations where the number of solutions is the problem.

A tangent is a line which only intersects at one point. (Note: you may not solve in this topic using Calculus methods, even if that is easier.)

Find k so that $y = 2x + k$ is a tangent to $y = x^2 + 3x$

$2x + k = x^2 + 3x$	$y = y$
$0 = x^2 + 3x - 2x - k$	quadratic, so all terms to one side
$x^2 + x - k = 0$	simplifying $a = 1$, $b = 1$, $c = {}^-k$

we are told there is one solution to this equation, so need $b^2 - 4ac = 0$

$1^2 - 4 \times 1 - k = 0$

$1 = {}^-4k$

$k = {}^-0.25$

If you are asked for a range of values to make something true, then solve for the limit values then work out the range using graphical or other methods.

Find k so that $y = kx + 2$ intersects $y = x^2 + x + 6$

$x^2 + x + 6 = kx + 2$	$y = y$
$x^2 + x + - kx + 4 = 0$	quadratic, so all terms to one side
$x^2 + (1 - k)x + 4 = 0$	simplifying $a = 1$, $b = (1 - k)$, $c = 4$

to find the limit values, we need $b^2 - 4ac = 0$

$(1 - k)^2 - 4 \times 1 \times 4 = 0$

$k^2 - 2k - 15 = 0$	brackets expanded and then simplified
$k = {}^-3$ or 5	solved by factorisation (or on graphics)

When $k = 0$, we get the line $y = 2$, which does not intersect $y = x^2 + x + 6$

So the values of k must be $k < {}^-3$ or $k > 5$

Different Variables

Sometimes, especially when dealing with real world systems, the variables used are not x and y.

The procedure doesn't change in any respect.

Merit students should get used to working with other variables.

The results can't be given in Cartesian co-ordinates. Note that both still have to be found.

Solve where $t = v^2 - 4v + 3$ meets $7v + 2t = 16$

step 1)	$t = {}^{-}3.5v + 8$	rearranged to give a term for t
step 2)	$v^2 - 4v + 3 = {}^{-}3.5v + 8$	$t = t$ when the lines intersect
step 3)	$v^2 - 4v + 3 + 3.5v - 8 = 0$	quadratic, so all terms to one side
	$v^2 + 0.5v - 5 = 0$	
	$v = {}^{-}2$ or 2.5	solved by factorisation (or on graphics)
step 4)	$t = {}^{-}3.5 \times {}^{-}2 + 8 = 15$	the t value for the first solution
	$t = {}^{-}3.5 \times 2.5 + 8 = {}^{-}0.75$	the t value for the second solution
	equations meet at $v = {}^{-}2$, $t = 15$ and at $v = 2.5$, $t = {}^{-}0.75$	

Different Systems of Equations

Not every situation can be covered in detail, but the methods work the same for all systems.

It is possible to find the intersection of two parabolas fairly easily.

Intersections of circles and hyperbolas with parabolas in general can have up to four solutions. You will not see them in Year 12.

Points to Note

1) Answers need to be given in terms of both x and y values, so care must be taken not to stop until both are found.

2) There are no marks for style, but you need to make your working clear so the marker can follow it. It helps to work down the page in a systematic way step by step. This also reduces student errors.

3) Learning the methods for the various situations by heart allows you to concentrate on the problem you face, instead of having to focus on trivial details of manipulation.

4) Most solutions can be quickly checked graphically on your graphics calculator. If not, then the values of the variables can be placed into both equations to see if they are both true for those values.

5) It is better to calculate and answer in fractions rather than round decimals, as it gives more accurate answers.