Question One Based on Team Solutions 2009 and 2010

(a) $p = 3 + 4i$ and $q = 2 - 3i$ are two complex numbers.

(i) Find $\frac{|p|}{q}$ in exact rectangular form.

(ii) Write $\bar{q}$ in polar form.

(b) $(x + 3 + 4i)$ is one root of a quadratic. Give that quadratic in polynomial form.

(c) $u = k \text{ cis } \left(\frac{9\pi}{10}\right)$ and $v = k \text{ cis } \left(\frac{-\pi}{10}\right)$ are two complex numbers.

(i) Find $u.v$ in polar form, with the argument in terms of π.

(ii) Find u^5 in polar form and show it is a purely imaginary number.

(d) Find the values of z for which $z^5 = 1 + i$

(e) Show that $z^2 + 4$ is a factor of $z^4 - 4z^3 + 9z^2 - 16z + 20$.

From that, or otherwise, find all roots of the equation $z^4 - 4z^3 + 9z^2 - 16z + 20 = 0$.

Question Two

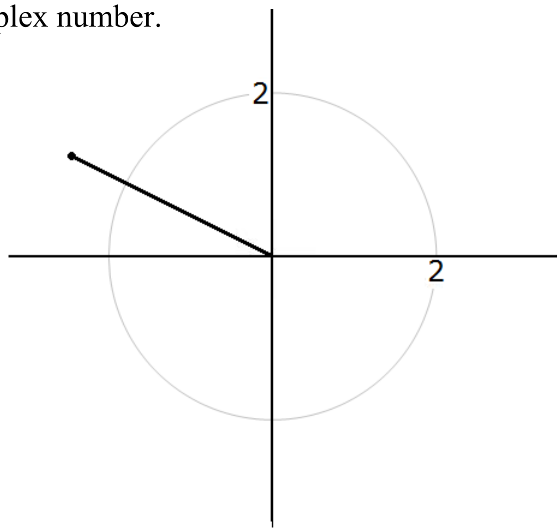
(a) Solve the equation $3z^2 = z - 2$, leaving your answer in surd form.

(b) Solve the equation $3^{2x + k} = 20$ for x in terms of k.

(c) Write $\frac{2 + \sqrt{20}}{4 - \sqrt{5}}$ in the form $a + b\sqrt{c}$

(d) On the Argand diagram to the right is shown a complex number.

Indicate approximately on the diagram that number's square root(s).



(e) Solve the equation $\sqrt{x} + 6 = 2x$

(f) $z = 1 + i$ is a root of the equation $z^3 - 8z^2 + kz - 12 = 0$. Find the value of k.

(g) z is a complex number so that $|z| = 1$ and $\frac{z-i}{z-1}$ is purely imaginary.

Show that z lies on a circle in the Argand plane, and find the equation of that circle.

Question One

(a) $p = 3 + 4i$ and $q = 2 - 3i$ are two complex numbers.

(i) Find $\frac{|p|}{q}$ in exact rectangular form.

$= \frac{\sqrt{3^2+4^2}}{2-3i} = \frac{\sqrt{25}}{(2-3i)} \frac{(2+3i)}{(2+3i)} = \frac{10+15i}{4+6i-6i-9i^2} = \mathbf{\frac{10+15i}{13}}$ A

(ii) Write $\bar{q}$ in polar form.

$\bar{q} = 2 + 3i$ $|q| = \sqrt{2^2+3^2} = \sqrt{13}$ $\arg(q) = \tan^{-1}(\frac{3}{2}) = 0.9827$ = **3.606 cis 0.9828** **A**

(b) $(x + 3 + 4i)$ is one root of a quadratic. Give that quadratic in polynomial form.

$(x + 3 + 4i)(x + 3 - 4i) = x^2 + 3x - 4xi + 3x + 9 - 12i + 4x\,i + 12i - 16i^2 = \mathbf{x^2 + 6x + 25}$ **A**

(c) $u = k \text{ cis} \left(\frac{9\pi}{10}\right)$ and $v = k \text{ cis} \left(\frac{-\pi}{10}\right)$ are two complex numbers.

(i) Find $u.v$ in polar form, with the argument in terms of π.

$k \text{ cis} \left(\frac{9\pi}{10}\right) \times k \text{ cis} \left(\frac{-\pi}{10}\right) = (k \times k) \text{ cis} \left(\frac{9\pi}{10} + \frac{-\pi}{10} \right) = \mathbf{k^2 \text{ cis} \left(\frac{4\pi}{5} \right)}$ **A**

(ii) Find u^5 in polar form and show it is a purely imaginary number.

$u^5 = (k \text{ cis} \left(\frac{9\pi}{10} \right)^5 = k^5 \text{ cis} \left(5 \times \frac{9\pi}{10} \right) = k^5 \text{ cis} \left(\frac{45\pi}{10} \right) = k^5 \text{ cis} (4.5\pi)$

But each 2π is back to where the number starts

$u^5 = k^5 \text{ cis} (0.5\pi) = k^5 (\cos (0.5\pi) + i \sin (0.5\pi)) = k^5 (0 + i \times 1) = \mathbf{k^5 i}$ M

$k^5 i$ has no real portion. It is purely imaginary.

(d) Find the values of z for which $z^5 = 1 + i$

$z^5 = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$ $\quad |z| = \sqrt{1^2 + 1^2} = \sqrt{13}$ $\quad \arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = 0.25\pi$

By De Moivre's Theorem $z = \sqrt[5]{\sqrt{2}}\ \text{cis}\left(\frac{\pi}{4} \div 5\right)$

$z_1 = = \sqrt[10]{2}\ \text{cis}\left(\frac{\pi}{20}\right) \quad = 1.0717\ \text{cis}\ (0.1571) \quad = 1.0586 + 0.1677i$

Other solutions are $+ 2\pi/5$ more

$z_2 = = \sqrt[10]{2}\ \text{cis}\left(\frac{9\pi}{20}\right) \quad = 1.0717\ \text{cis}\ (0.1571) \quad = 0.1677 + 1.0586i$

$z_3 = = \sqrt[10]{2}\ \text{cis}\left(\frac{17\pi}{20}\right) \quad = 1.0717\ \text{cis}\ (1.4137) \quad = {}^-0.9550 + 0.4866i$

$z_4 = = \sqrt[10]{2}\ \text{cis}\left(\frac{25\pi}{20}\right) \quad = 1.0717\ \text{cis}\ (2.6704) \quad = {}^-0.7579 - 0.7579i$

$z_5 = = \sqrt[10]{2}\ \text{cis}\left(\frac{33\pi}{20}\right) \quad = 1.0717\ \text{cis}\ (3.9270) \quad = 0.4866 - 0.9550i$ **M**

(any form acceptable, but must have all answers for **M**)

(e) Show that $z^2 + 4$ is a factor of $z^4 - 4z^3 + 9z^2 - 16z + 20$.

From that, or otherwise, find all roots of the equation $z^4 - 4z^3 + 9z^2 - 16z + 20 = 0$.

$$
\begin{array}{r|l}
 & z^2 \quad - 4z \quad + 5 \\
z^2 + 4 & z^4 - 4z^3 + 9z^2 - 16z + 20 \\
 & \underline{-z^4 \qquad\quad - 4z^2} \\
 & \quad - 4z^3 + 5z^2 - 16z + 20 \\
 & \quad \underline{+ 4z^3 \qquad\quad + 16z} \\
 & \qquad\qquad 5z^2 \qquad + 20 \\
 & \qquad\qquad \underline{- 5z^2 \qquad - 20} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

So $z^4 - 4z^3 + 9z^2 - 16z + 20 = (z^2 + 4)(z^2 - 4z + 5)$ which means $z^2 + 4$ is a factor

Solving the two quadratics separately:

$z^2 + 4$ has roots $2i$ and ${}^-2i$ $\quad z^2 - 4z + 5$ has roots $2 + i$ and $2 - i$ which are the four roots

$2i$, ${}^-2i$, $2 + i$ and $2 - i$ are the four roots of $z^4 - 4z^3 + 9z^2 - 16z + 20$ **E**

(Must give as roots, not factors for E)

alternatively,

$z^2 + 4$ has factors $2i$ and ${}^-2i$. $(2i)^4 - 4(2i)^3 + 9(2i)^2 - 16(2i) + 20 = 0$, so $2i$ must be a factor.

$({}^-2i)^4 - 4({}^-2i)^3 + 9({}^-2i)^2 - 16({}^-2i) + 20 = 0$, so ${}^-2i$ is also a factor.

$(z^2 + 4)(az^2 + bz + c) = z^4 - 4z^3 + 9z^2 - 16z + 20$ can be solved by coefficient matching

$4c = 20 \Rightarrow c = 5$, $az^4 = z^4 \Rightarrow a = 1$ and ${}^-16z = 4bz \Rightarrow b = {}^-4$ and then solve as before

Question Two

(a) Solve the equation $3z^2 = z - 2$, leaving your answer in surd form.

$3z^2 - z = {}^-2$ $\quad \Rightarrow z^2 - \frac{1}{3}z = \frac{-2}{3}$ $\quad \Rightarrow (z - \frac{1}{6})^2 - \frac{1}{36} = \frac{-24}{36}$

$\Rightarrow (z - \frac{1}{6})^2 = \frac{-23}{36}$ $\quad \Rightarrow z - \frac{1}{6} = \pm\sqrt{\frac{-24}{36}}$ $\quad \Rightarrow \mathbf{z = \frac{1}{6} \pm \frac{\sqrt{23}}{6} i}$ **A**

(b) Solve the equation $3^{2x + k} = 20$ for x in terms of k.

$(2x + k) \log 3 = \log 20$ $\quad 2x + k = \frac{\log 20}{\log 3}$ $\quad 2x + k = 2.7268$

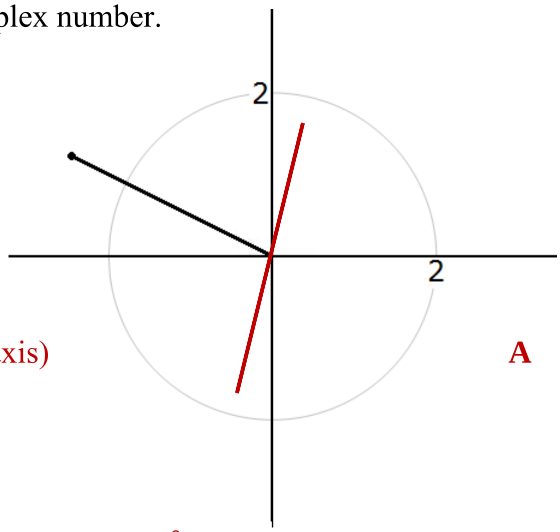
$2x = 2.7268 - k$ $\quad \mathbf{x = 1.363 - 0.5k}$ **A**

(c) Write $\frac{2+\sqrt{20}}{4-\sqrt{5}}$ in the form a + b√c

$= \frac{2+\sqrt{20}}{4-\sqrt{5}} \frac{4+\sqrt{5}}{4+\sqrt{5}} = \frac{8+2\sqrt{5}+4\sqrt{20}+\sqrt{100}}{16+4\sqrt{5}-4\sqrt{5}-\sqrt{25}} = \frac{18+2\sqrt{5}+4\sqrt{4}\sqrt{5}}{11} = \frac{18}{11} + \frac{10}{11}\sqrt{5}$ **A**

(d) On the Argand diagram to the right is shown a complex number.

Indicate approximately on the diagram that number's square root(s).



$\sqrt{r \operatorname{cis}\theta} = \sqrt{r} \operatorname{cis}\frac{\theta}{2}$

Line must be less than 2 long, but more than 1.

Angle must be about half original (from positive x axis) **A**

(e) Solve the equation $\sqrt{x} + 6 = 2x$

$\sqrt{x} = 2x - 6$ $\quad (\sqrt{x})^2 = (2x - 6)^2$ $\quad x = 4x^2 - 24x + 36$

$4x^2 - 25x + 36 = 0$

Solving gives $x = 4$ and 2.25

Verifying shows that $\sqrt{4} + 6 = 2 \times 4$, but that $\sqrt{2.25} + 6 \neq 2 \times 2.25$

Solution is $x = 4$ **M**

(f) z = 1 + i is a root of the equation $z^3 - 8z^2 + kz - 12 = 0$. Find the value of k.

if $z = 1 + i$ is a root, then by the Factor Theorem $f(1+i) = 0$

$(^-1 - i)^3 - 8(^-1 - i)^2 + k(^-1 - i) - 12 = 0$

$(^-2 + 2i) - 8(2i) + k(1 + i) - 12 = 0$

Grouping real and imaginary portions: $^-2 + ki - 12 = 0$ and also $2i - 16i + ki = 0$

Both these solve to give $k = 14$ **M**

Alternatively:

$(z - 1 - i)(z - 1 + i)(az + b) = z^3 - 8z^2 + kz - 12$ factors from – root and – conjugate

$(z^2 - 2z + 2)(az + b) = z^3 - 8z^2 + kz - 12$ Coefficient matching gives $a = 1$ and $b = {}^-6$

$(z^2 - 2z + 2)(z - 6) = z^3 - 8z^2 + 14z - 12$ $\Rightarrow k = 14$

(g) z is the complex number $x + yi$. $\frac{z-i}{z-1}$ is purely imaginary.

Show that z lies on a circle in the Argand plane, and find the equation of that circle.

$$\frac{z-i}{z-1} = \frac{x+yi-i}{x+yi-1} = \frac{x+(y-1)i}{(x-1)+yi}$$

$$= \frac{x+(y-1)i}{(x-1)+yi}\,\frac{(x-1)-yi}{(x-1)-yi} = \frac{x(x-1) - xyi + (y-1)(x-1)i - (y-1)yi^2}{(x-1)(x-1) - y^2i^2}$$

Ignoring the bottom line now, since it is entirely real terms, sticking to the top line

$= x^2 - x - xyi + xyi - yi - xi + 1i - y^2i^2 + yi^2$

$= x^2 - x + y^2 - y + (1 - x - y)i$

But we are told this expansion is purely imaginary, so the real part must be zero

$\Rightarrow x^2 - x + y^2 - y = 0$

$(x - ½)^2 - ¼ + (y - ½)^2 - ¼ = 0$

$(x - ½)^2 + (y - ½)^2 = ½$ is the equation of a circle, centre (½ , ½), radius √½ **E**

N1, N2, A3, A4 = for each Achieved question

M5, M6 = for each Merit

E7 = for incomplete Excellence

E8 = for complete Excellence

Final Grade Score Boundaries: 0–5 = Not Achieved, 6–8 Achieved, 9–12 Merit, 13+ Excellence