Algebra Year 10

Introduction

In Algebra we do Maths with numbers, but some of those numbers are not known. They are represented with letters, and called “unknowns”, “variables” or, most formally, “literals”.

All the normal rules of Mathematics apply. **All**.

When we write $a + b$, we are adding the number a to the number b. Since we know that the order in which numbers are added does not matter, so we can write $a + b = b + a$ without knowing what a and b are.

But the order of subtraction does matter. For example, $4 - 6 = -2$ while $6 - 4 = 2$. So we cannot rearrange a subtraction like an addition: $a - b \neq b - a$ ($\neq$ means “does not equal”).

It is often useful to consider known numbers when trying to decide whether what you are doing in Algebra is correct. If it is not true with known values, then it is not true with unknown values either.

Language

A “term” is any numbers and variables written together to indicated multiplication.

An “expression” is a term or series of terms.

An “equation” is a when some expression is written to equal a number or another expression,

$15 + 6x = 3x$ is an equation, where the expression $15 + 6x$ is equal to the term $3x$.

You do not need to be able to use this language, but it helps to recognise it.

Conventions

It is normal in Algebra to leave out the multiplication signs. We write xy instead of $x \times y$, and we write $4x$ for $4 \times x$.

We never write the multiplying number after the unknowns: $x \times 6 = 6x$ (never $x6$).

It is more normal to write any divisions as a fraction. We write $\frac{x}{2}$ for $x \div 2$.

There is no need to put a leading “1” in front of a term. We write xy rather than $1xy$.

There is no need to put a power of 1 for a term. We write x rather than x^1.

It is traditional, but not necessary, to put unknowns in alphabetical order.

Blue Beta: 22.1, 22.2
New Beta: 7.01, 7.02

Addition and Subtraction

Repeated addition is the same as multiplication. This means in effect **we add the leading numbers**.

$x + x + x \;=\; 3x$

$2y + 4y \;=\; y + y \;+\; y + y + y + y \;=\; 6y$

We cannot add terms unless they contain **exactly** the same unknowns. This is usually described as **we can only add "like terms"**.

$3x + 5y$ can never be added, because they are not "like".

$4x^2$ and $5x$ can never be added, since different powers make terms un-"like".

When we have a string of terms we can separately add the like ones (by adding up the leading multipliers) but we cannot combine the non-like.

$$3x + 5y + 8x + 2y \;=\; (3 + 8)\,x + (5 + 2)\,y \;=\; 11x + 7y$$

Subtraction is done exactly like addition, except that the number parts of the terms are subtracted.

$$4y - 2y \;=\; y + y + y + y - y - y \;=\; 2y$$

As with addition, only "like" terms can be subtracted.

$$4x + 5y - 2y \;=\; 3x + (5 - 2)\,y \;=\; 4x + 3y$$

As with ordinary arithmetic, if a bigger number is subtracted from a smaller one, the result will be negative.

$$3y - 12y \;=\; (3 - 12)\,y \;=\; {}^{-}9y$$

The subtraction sign applies only to the term immediately after it.

$3f - 2k + 12$ has only one subtraction: the $2k$ is subtracted but the 12 is added.

A subtraction can be turned into an addition of a negative and then rearranged as usual for addition. But it is entirely incorrect to move terms around in a subtraction without keeping the negative sign with the appropriate term.

$3f - 2k$ can be rewritten as ${}^{-}2k + 3f$ but cannot be rewritten as $2k - 3f$.

Blue Beta: 23.2
New Beta: 8.02, 8.03

Multiplying

Any two terms can be multiplied. There is no equivalent of the "like terms" needed for addition and subtraction.

An unknown multiplied by itself is indicated by marking it with an appropriate power. This means **multiplying terms together is equivalent to adding the powers**.

$$x \times x \times x = x^3$$

$$y^2 \times y^4 = y \times y \times y \times y \times y \times y = y^6$$

When we multiply complex terms, **we multiply each component** and simplify those that can be simplified.

$$3x \times 5y = 3 \times x \times 5 \times y = 15xy$$

$$4x^2 \times 5x = 4 \times x \times x \times 5 \times x = 20x^3$$

Many people multiply by adding the powers of the unknowns. While this works it is important to remember that if the unknown has no power marked then it is 1, not 0.

$$2x \times 5x \times 8x^2 = 2x^1 \times 5x^1 \times 8x^2 = 80x^{1+1+2} = 80x^4$$

If the numbers happen to be negative they are multiplied as normal.

$$^{-}2k \times 3f = {}^{-}6kf$$

It is vital to remember what the power of an unknown represents. As it is the unknown multiplied by itself a specific number of times it cannot be separated out from the unknown. Nor is it a simple multiplication (we use a number in front of the unknown for that): $x^2 = x \times x$ **not** $2 \times x$

Blue Beta: 23.1, 23.4 to 23.4
New Beta: 8.01, 8.05 to 8.07

Dividing

Division is usually best done by using fraction methods.

Any number divided by itself is equal to 1, and so cancels out. This applies to unknown numbers, as well as ordinary ones: We can write this since $\frac{x}{x} = 1$ for every x.

So **we can cancel out unknowns that occur both on the top and bottom**.

$$y^4 \div y^2 = \frac{y \times y \times y \times y}{y \times y} = \frac{\cancel{y \times y} \times y \times y}{\cancel{y \times y}} = y^2$$

When we divide complex terms, **we divide each component** and simplify where possible.

$$15xy \div 10y = \frac{15 \times x \times y}{10 \times y} = \frac{3 \times x \times \cancel{y}}{2 \times \cancel{y}} = \frac{3x}{2}$$

The number component of any division is cancelled just like a fraction:

$$5x \div 10 = \frac{5x}{10} = \frac{1 \times x}{2} = \frac{x}{2} \quad \text{(since } 5 \div 10 = ½\text{)}$$

A common cause of error is improperly swapping $a \div b$ with $b \div a$, which is why doing it as a fraction is better. If cancelling results in a number being on the bottom line, then it must stay there:

$$2x^2 \div 6x = \frac{2x^2}{6x} = \frac{1 \times x \times \cancel{x}}{3 \times \cancel{x}} = \frac{x}{3} \quad \text{(which cannot be made into } = 3x\text{)}$$

Likewise, if the power of the unknown in the divisor is larger, then the answer will have the unknown on the bottom line. A "1" is left on the top line if everything else cancels out

$$x^2 \div x^3 = \frac{\cancel{x} \times \cancel{x}}{x \times \cancel{x} \times \cancel{x}} = \frac{1}{x} \quad \text{(which cannot be made into } = x\text{)}$$

Many people divide by subtracting the powers of the unknowns. While this works, it is important to make sure the subtraction is done the right way round. The result may be a negative number.

$$x^2 \div x^3 = x^{2-3} = x^{-1} \quad \text{(which cannot be turned into } = x^{3-2} = x\text{)}$$

If the numbers happen to be negative they are divided as normal. You will rarely be asked this.

$$^{-}2fk \div {}^{-}4f = \frac{-2 \times f \times k}{-4 \times f} = \frac{1 \times \cancel{f} \times k}{2 \times \cancel{f}} = \frac{k}{2}$$

The components of an algebraic fraction can be separated out, but anything on the bottom line stays there and a "1" must be used to hold values on the bottom line.

$$\frac{k}{2} = \frac{1}{2}\frac{k}{1} = \frac{1}{2}k \qquad \text{and} \qquad \frac{2}{x} = \frac{2}{1}\frac{1}{x} = 2\,\frac{1}{x} \quad (= 2x^{-1})$$

Blue Beta: 23.8
New Beta: 8.06 to 8.07

Expanding

Removing brackets is called "expansion". Following BEDMAS it has the highest priority in order of what to do.

Any term outside the bracket is multiplied by every term inside the bracket.

$4\,(x + 5) \;=\; 4 \times x + 4 \times 5 \;=\; 4x + 20$

Any negative is part of the following term, whether inside or outside the bracket.

$^{-}4\,(x + 10) \;=\; {^{-}4} \times x + {^{-}4} \times 10 \;=\; {^{-}4x} - 40$

$5\,(x - 6) \;=\; 5 \times x + 5 \times {^{-}6} \;=\; 5x - 30$

The term outside the bracket can include unknowns.

$x\,(x - 2) \;=\; x \times x + x \times {^{-}2} \;=\; x^2 - 2x$

If the terms are complex the procedure is just the same.

$^{-}5x\,(2x + 6) \;=\; {^{-}5x} \times 2x + {^{-}5x} \times 6 \;=\; {^{-}10x^2} + {^{-}30x} \;=\; {^{-}10x^2} - 30x$

Care must be taken with negatives. In particular, a negative outside multiplied by one inside will give a positive product.

$-\,4(k - 2) \;=\; {^{-}4} \times k + {^{-}4} \times {^{-}8} \;=\; {^{-}4k} + 8$

It can help to draw in the multiplication links.

$5\,(\,x + 2\,) \;=\; 5x + 10$

If two brackets are expanded, then the results may be simplified, if there are like terms.

$5(k - 5) + 4(k - 2) \;=\; 5k - 25 + 4k - 8 \;=\; 9k - 33$

The most common errors are:

- Forgetting to *multiply* the term in front by each term,
- Not leaving the result as a series of unlike terms separated by + or –, but attempting to simplify them instead,
- Difficulty with negatives, especially multiplying two negatives.

Blue Beta: 24.1 to 24.4
New Beta: 9.01 to 9.03

Factorising

Factorising is the process of taking the common factors out of a series of terms and leaving the resulting terms in brackets. It is the opposite of expanding.

The highest common factor is placed outside the brackets.

$4x$ and 20 have a highest common factor of 4, so $4x + 20 = 4(x + 5)$

An unknown can be a common factor.

x^2 and $3x$ have a common factor of x, so $x^2 + 3x = x\,(x + 3)$

The highest common factor might not divide both terms entirely.

$$8x + 12 = 4\,(2x + 3)$$

The highest common factor can be a complex term. It is often easiest to do such factorising in two stages.

$$10x^2 + 30x = 5x\,(2x + 6)$$

If the highest common factor is one of the terms, then its place in the brackets will be held by a 1.

$$6x + 6 = 6\,(x + 1)$$

If all the terms inside the bracket are negative, then the common factor will be negative. Care must be taken to put the correct signs inside the bracket when doing this.

$$-4k - 8 = {}^{-}2\,(k + 4)$$

If a question asks you to factorise, you must assume that you will fully factorise the expression.

$4x + 8 = 2\,(2x + 4)$ is true, but not enough. You need to give $= 4\,(x + 2)$

It is often easiest to factorise by treating it as the reverse of expansion. Once the highest common factor is found, the terms inside the bracket are found by trial and error using expansion.

Students need to understand simple factorising well before attempting to factorise anything difficult.

Blue Beta: 24.5 to 24.8
New Beta: 9.04 to 9.07

Expanding two brackets

Quadratics, as pairs of brackets are called, require care.

> **Every term in the first bracket is multiplied by every term inside the second bracket**.
>
> $(x + 4)\ (x + 5)\ =\ x^2 + 5x + 4x + 20$
>
> Then any "like" terms (usually the middle two) have to be combined.
>
> $x^2 + 5x + 4x + 20\ =\ x^2 + 9x + 20$

It helps to draw in the multiplication links. *FOIL = First – Outside – Inside – Last*

$$(\ x + 5\ )\ (\ x\ +\ 2\ ) \qquad = \ 5x + 10$$

If the terms are negative the procedure is just the same.

$$(x - 4)(x + 6) \qquad = x^2 + 6x + {}^{-}4x + {}^{-}24 \qquad = x^2 + 2x - 24$$

Extra care are must be taken with two negative signs inside the brackets. The end number will be positive, as it will be a negative times a negative.

$$(x - 3)(x - 2) \qquad = x^2 + {}^{-}2x + {}^{-}3x + 6 \qquad = x^2 + {}^{-}5x + 6$$

Nothing changes if the terms are more complicated or in a different order, although great care is needed in the multiplication stage and finding the like terms.

$$(2x - 4)(6 + x) \qquad = 12x + 2x^2 + {}^{-}24 + {}^{-}4x \qquad = 2x^2 + 8x - 24$$

The most common errors are:

- Forgetting to *multiply* all the way through the first step (adding the last two is common)
- Forgetting to simplify the like terms,
- Difficulty with negatives, especially adding a negative and positive term in the middle.

Blue Beta: 26.4 to 26.7
New Beta: 11.04 to 11.07

Factorising two brackets

Factorising quadratics requires confidence in expanding them first.

> The quadratic should be written in the order of the powers of x.
>
> $10x + x^2 + 24$ *is written as* $x^2 + 10x + 24$
>
> The pairs of factors that multiply to give the last term are found.
>
> $+ 24$ *has factor pairs of* 1×24, 2×12, 3×8 and 4×6
>
> The middle terms is compared to those adding those factor pairs.
>
> $+ 10 = 6 + 4$ (*but is not equal to* $1 + 24$, $2 + 12$ or $3 + 8$)
>
> The pair of numbers found above are those in the brackets of the factorised form.
>
> $10x + x^2 + 24 = (x + 6)(x + 4)$

The order of brackets is not relevant.

$$(x + 6)(x + 4) = (x + 4)(x + 6)$$

If the last terms is negative, then one of the factor pair is negative. If the bigger factor is the positive one, then the middle term is positive, if the bigger factor is negative, then the middle term is negative.

$x^2 + 4x - 21 = (x + 7)(x - 3)$ (*as* $7 \times {}^-3 = {}^-21$ *and* $7 + {}^-3 = 4$)

compared to

$x^2 - 4x - 21 = (x - 7)(x + 3)$ (*as* ${}^-7 \times 3 = {}^-21$ *and* ${}^-7 + 3 = {}^-4$)

If the end term is positive, but the middle one is negative, then both factors will be negative.

$x^2 - 9x + 20 = (x - 4)(x - 5)$ (*as* ${}^-4 \times {}^-5 = 20$ *and* ${}^-4 + {}^-5 = {}^-9$)

There is a special case when the middle term is not present.

$x^2 - 16 = (x - 4)(x + 4)$

You will not be asked to factorise more complicated brackets at Year 10.

Blue Beta: 26.8 to 26.11
New Beta: 11.08 to 11.12

Which sort of factorising?

A lot of students factorise well when they know what sort of factorising to do. The trick is to recognise the patterns when they are mixed up.

If every term in the expression has a number in front, then any common factor can be removed:

$12x + 24 = 12(x + 2)$

If the terms have a common unknown, then it can be removed as a common factor:

$x^2 + 24x = x(x + 24)$

If the terms have nothing in common, and include an x^2, then they will be a quadratic:

$10x + x^2 + 24 = (x + 6)(x + 4)$

The pattern "x^2 – something" gives the $(x + a)(x – a)$ form:

$x^2 – 16 = (x – 4)(x + 4)$

Sometimes once one factor is removed another still remains. Factorisation can be a two step process as a result.

$4x^2 – 8x = 4(x^2 – 2x) = 4x(x – 2)$

Once confident with the ability to factorise in each way, it is always best to practice with mixed examples. Unfortunately text books are not good at providing mixed examples.

Substituting

Substitution is the process of replacing unknown values with known ones.

The unknowns in an expression or equation are replaced with the known values.

if $a = 3$ and $b = 5$, then: $a\,(b + 2) = 3 \times (5 + 2) = 21$

All the normal rules of arithmetic apply at all times, and in particular BEDMAS.

Lines on fractions are considered to be bracketed and must be done first.

$\text{Av} = \frac{a + b}{2}$ if $a = 7$ and $b = 9$

$\text{Av} = \frac{7 + 9}{2} = \frac{16}{2} = 8$

The result needs to be fully calculated to give a **single** numerical answer.

$e^2 + 30e$ if e = 2

$2^2 + 30 \times 2 = 4 + 60 = 64$

The unknown replaced must be replaced fully with the value. If that value is negative then care is needed to make sure that this is done properly. Usually it is best to bracket negatives when doing this on a calculator.

$e^2 + 30e$ if $e = -3$

$(-3)^2 + 30 \times ({}^-3) = 9 + {}^-90 = {}^-81$

Blue Beta: 22.3 to 22.5
New Beta: 7.03 to 7.05

Solving

Solving is the process of finding the value (or values) of an unknown that make an equation true.

The solution will always end in the form: x = *something*

What we do to one side of an equation, we must do to the other, so as to keep it balanced.

$$x + 4 = 9$$
$$x + 4 - 4 = 9 - 4$$
$$x = 5$$

To remove a term from one side of an equation, and so simplify it, we do the opposite operation.

The opposite of multiplication is division:

$5x = 10$

to find what x = , we need to remove the 5 ×

so: $\frac{5x}{5} = \frac{10}{5}$ dividing both sides by 5

so: $x = 2$ answer

The opposite of division is multiplication:

$\frac{x}{4} = 2$

to find what x = , we need to remove the ÷ 4

so: $\frac{4x}{4} = 2 \times 4$ multiplying both sides by 4

so: $x = 8$ answer

The opposite of addition is subtraction:

$x + 5 = 8$

to find what x = , we need to remove the + 5

so: $x + 5 - 5 = 8 - 5$

so: $x = 3$

The opposite of subtraction is addition:

$x - 2 = 3$

to find what x = , we need to remove the – 2

so: $x - 2 + 2 = 3 + 2$

so: $x = 5$

Blue Beta: 25.1 to 25.4
New Beta: 10.01 to 10.04

Solving equations with multiple terms

Generally solving an equation involves multiple steps.

We can move unknown terms in the same manner as number terms.

$$6x = 5x + 2$$
$$6x - 5x = 5x - 5x + 2$$
$$x = 2$$

To solve multiple step equations **we first need to group all the unknown terms on one side of the equation and all the number terms on the other**.

$$6x + 4 = 2x + 8$$
$$6x + 4 - 4 = 2x + 8 - 4$$
$$6x = 2x + 4$$
$$6x - 2x = 2x - 2x + 4$$
$$4x = 4$$
$$x = 1$$

Any negative sign stays with the term that follows.

$$4x = 2x - 3$$
so: $4x - 2x = 2x - 2x - 3$
so: $2x = -3$
so: $x = -1.5$

If there are brackets, then it is best to remove the brackets first, then solve as normal.

$3(x - 3) = 6$ becomes $3x - 9 = 6$

When using any operation it is vital that it applies to **all** of both sides

$$\frac{x}{4} = x + 5$$
so: $\frac{4 \times x}{4} = 4\,(x + 5)$
so: $x = 4x + 20$

Students **must** resist the temptation to "solve" problems by working backwards using just the numbers in reverse order. It works for simple problems but leaves students unable to move on to harder ones.

Blue Beta: 25.5 to 25.15
New Beta: 10.05 to 10.14

Solving Inequations

An inequation is like an equation, but with $<$, $\leq$, $>$, or $\geq$ where a = would normally be.

The same basic techniques are used as with solving equations. In general the sign stays the same.

$$x + 4 > 9$$
$$x + 4 - 4 > 9 - 4$$
$$x > 5$$

If the sides of the inequation are swapped, the sign must be reversed to make it true.

If $4 > x$ *then* $x < 4$

If the equation is multiplied or divided by a negative, the sign must be reversed.

$$-4x < 9$$
$$-4x \div -4 > 9 \div -4 \qquad \text{(\textit{sign swapped as dividing by negative})}$$
$$x > -2.25$$

The sign is only swapped if the process involves a multiplication or division by a negative. Nothing changes just because the answer is negative.

$$x + 3 > 1$$

so: $x + 3 - 3 > 1 - 3$

so: $x > -2$

Many find it easier to solve any inequations in such a way that the *x* term is never negative.

$$2x + 3 > 6x + 1$$

so: $2x - 2x + 3 > 6x - 2x + 1$

so: $3 > 4x + 1$

so: $4x + 1 - 1 < 3 - 1$ etc

When considering which step to take next in an inequation, it will be exactly what one would do in an equation.

Blue Beta: 25.16
New Beta: 11.08 to 11.12

Solving Quadratics

A quadratic equation must be solved in a totally different manner from linear equations.

> The basic concept is that if $ab = 0$, then either $a = 0$ or $b = 0$, so that if two bracketed terms multiplied together = 0, then either the first bracket or the second bracket = 0.
>
> $(x + 4)(x + 5) = 0$
>
> *means either* $(x + 4) = 0$ or $(x + 5) = 0$
>
> *so either* $x = {}^{-}4$ or $x = {}^{-}5$
>
> Quadratics must be factorised in order to be solved.
>
> $x^2 - 4x - 21 = 0$
>
> $(x - 7)(x + 3) = 0$
>
> $x = 7$ or $x = {}^{-}3$

If the terms are spread across the equals sign they must be grouped first by rearrangement.

$x^2 - 2x = 24$

so: $x^2 - 2x - 24 = 0$

so: $(x - 6)(x + 4) = 0$

so: $x = 6$ or $x = -4$

Solving quadratics requires recognising the x^2 term, and that they need to be factorised first. The process needs to be properly memorised, because students cannot recall it otherwise under test conditions.

Blue Beta: none
New Beta: 11.14 to 11.17

Putting it together

Generally students find that the real difficulties come when they have to use their skills in situations where the questions are not all of the same type.

It therefore makes sense, once the basic understanding has been reached, to practice basic skills only in situations where the types of questions are mixed. If students only practice in situations where it is clear which skill they will be using before they start they will not build the ability to distinguish which methods to use as quickly.

At first practice should involve just ×, ÷, + and – . Sadly, very few textbooks give any practice of questions with the operations mixed up.

Once that is routine, practice should involve expanding brackets mixed with other operations.

To bear in mind, when doing practice with mixed operations:

- BEDMAS still applies at all times.
- Only addition and subtraction require like terms.
- The powers of unknowns only change with multiplication and division.
- x is the short way of writing $1x^1$
- Answers with like terms should be simplified, but resist the temptation to keep on going.
- Extra care must be taken with negatives: they always stay with the following term.
- Two negatives multiplied give a positive, but two negatives added give a negative.

Multiple step problems should be solved with clearly written multiple steps. It is a false economy to try to do all the working in one line. It leads to errors (especially with negatives) and makes it confusing for students to try and remember the order of steps. Clearly separate each process.

Students need to very clearly sort out when a question is a quadratic and when it is linear. They require entirely different methods, which need to be practiced extensively. Students who do not practice are unable to distinguish between the different types in exams.

It can help to memorise a few basic facts to rely on:

$x + x = 2x$	*adding does not change the power*
$-x + -x = -2x$	*two negatives add to a negative*
$x \times x = x^2$	*multiplying does change the power*
$-x \times -x = +x^2$	*two negatives multiply to a positive*
$x \div x = 1$	*any number cancels when divided by itself*