Patterns and Graphing Year 10

While students may be shown various different types of patterns in the classroom, they will be tested on simple ones, with each term of the pattern an equal difference from the next.

Terminology

A pattern of numbers in order is technically a <u>sequence</u>.

The numbers listed are called the <u>terms</u>.

4, 6, 8, 10, … has a first term = 4, and a third term = 8.

When writing rules for patterns, it is traditional to use *n* as the unknown (in the way that traditionally we use *x* for algebra).

If a rule uses *n*, it is assumed that it starts at 1 and goes up through all the whole numbers.

If the pattern is formed by adding a constant number to the term before, that number is the <u>difference</u>.

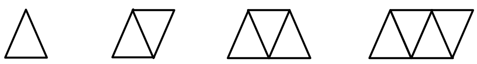
7, 17, 27, 37, … has a difference of + 10

Patterns where the differences are equal are called <u>linear</u>, because they are a line when graphed.

Students in Year 10 are not expected to be able to use the terminology yet. It is enough that they recognise it.

A lot of low level questions are posed with a pattern given as a set of diagrams. In often helps to convert the pictures to numbers for the harder parts.

e.g. how many triangle edges are needed to make the 10^{th} in this series?



converts to 3, 5, 7, 9,… which makes it much easier to get the answer (= 21)

Completing a Pattern

A linear pattern can be continued from a given example by finding the difference between terms and adding it to the next terms.

5, 8, 11, 14, … is going up by three each term, so the next terms are clearly 17 then 20

The numbers involve can be anything, so including negative and fractional terms and differences.

12, 7, 2, –3, –8, –13 … 2, 2½, 3, 3½, 4 , 4½, …

Students should be able to make a pattern when given the first term and the difference.

e.g. if the first term is 3, and the difference is +2, the sequence is: 3, 5 (3 + 2), 7 (5 + 2), …

Pattern questions can be given with the numbers in boxes. The same rules apply:

term	1	2	3	4	5	6
value	12	14	16	18		

is the same as: 12, 14, 16, 18, … and so the next boxes have 20 then 22 in them.

Pattern questions usually assume a fixed start point, but they can also be thought of as infinite in both directions. Earlier terms are found by subtracting the difference from the one before.

		15	17.5	20	22.5		

is the same as: … 15, 17.5, 20, 22.5 … which has a difference of +2.5

and so the first boxes have 10 and 12.5, and the last boxes have 25 and 27.5.

Sometimes the question will ask for a term in the sequence other than the one immediately following. This is better done by finding a rule, as otherwise it is too easy to make a mistake.

e.g. if we have to find the 20th term of the sequence 3, 6, 9, 12, 15, …

we can write the whole thing out, giving 60.

or we can work out that the difference is +3, and that each term is 3 × its position. So the 20th term is 3 × 20 = 60.

Students should be able to make a pattern from a rule, with the unknown in the rule starting from 1.

e.g. the rule n + 5, generates the pattern: 6 (1 + 5), 7 (2 + 5), 8 (3 + 5), 9 (4 + 5), …

e.g. the rule 2n – 1, generates the pattern: 1 (2 × 1 – 1), 3 (2 × 2 – 1), 5 (2 × 3 – 1), 7 (2 × 4 – 1), …

Finding a Rule

Students find this very difficult, especially the terminology. They have a couple of years to learn it.

The rule for linear patterns is the **difference × *n* + 0$^{\text{th}}$ term**.

For the sequence: 6, 10, 14, 18, 22, …

the rule is $4n + 2$

difference is +4   + 4

6,   10,   14,   18,   22,   …

0$^{\text{th}}$ term is 2   2 ← –4

The 0$^{\text{th}}$ term is the term that would be before the 1$^{\text{st}}$, found by taking the 1$^{\text{st}}$ term and subtracting the difference.

Students need to move beyond writing rules in the form "add two to the previous one". This is no longer acceptable when asked to write a rule.

The multiplier in the rule will be negative if the sequence is going down:

e.g. 20, 18, 16, 14, … has a difference of –2, and the 0 term would be $20 - {}^{-}2 = 22$

so its rule is: $-2n + 22$

The 0 term will be negative if the starting term is less than the difference:

e.g. 5, 15, 25, 35, … has a difference of +10, and the 0 term would be $5 - 10 = {}^{-}5$

so its rule is: $10n - 5$

It is vital not to have the 1$^{\text{st}}$ term as the + or – in the rule, and to remember that although the difference is + when writing the sequence in order, it is × in the formula.

The formula for the rule works because if a is the first term, and d is the difference, a sequence can be written as:

$a,\ a + d,\ a + d + d,\ a + d + d + d, \ldots$

$= a,\ a + 1d,\ a \times 2d, a + 3d, \ldots$

so each term is a + one less than the term number of ds, which is $= a + (n - 1)\,d$

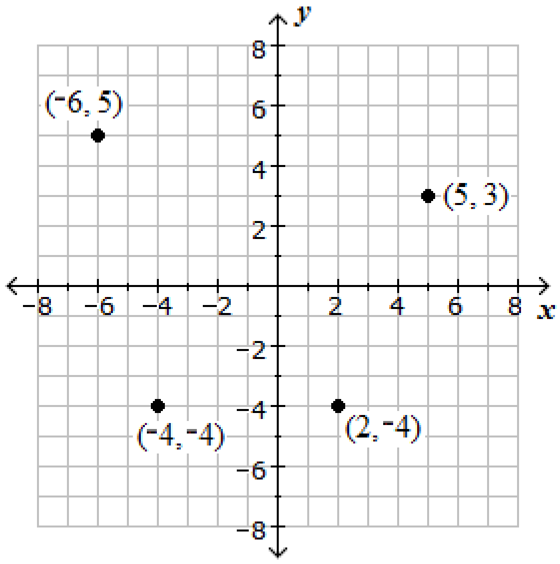
rearranging, $a + (n - 1)\,d = a - d + (n - 1)\,d + d = (a - d) + nd = 0^{\text{th}}$ term $+ nd$

Co-ordinate system

Students should be able to plot points quickly by the end of Year 10, without having to stop each time to remember which direction is *x* and which is *y*.

> Points are given in the form (*x*, *y*),
>
> *x* is the distance horizontally, with right as positive and left as negative. ("*x* is a cross")
>
> *y* is the distance vertically, with up as positive and down as negative. ("*y* to the sky")
>
> The scale is marked on the *x*-axis and *y*-axis (plural is axes). The point (0, 0) is called the origin.

Generally points are plotted on Cartesian grids, which extend infinitely in all directions.



In Maths the whole number are marked on the lines, not the spaces. Decimal values are marked the correct distance along, but it is easier if they are avoided as much as possible.

While there are some helpful mnemonics to let you remember that (4, 5) means four across then five up ("you need to run before you jump") students need to make it as natural as possible. Those who need to stop and think before they plot any points are badly disadvantaged compared to those who do it automatically.

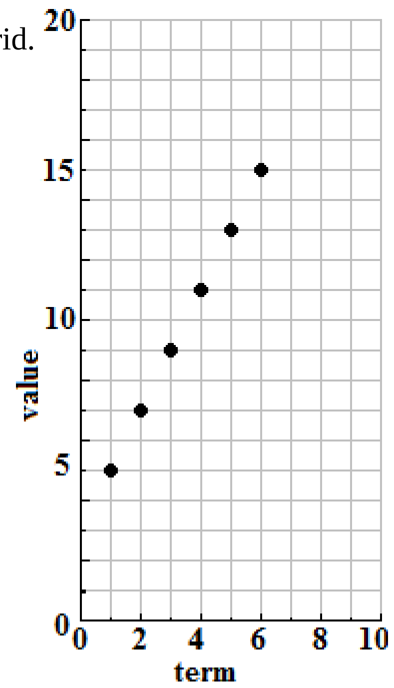
Graphing a Pattern

When graphing any relationship – given by some pattern, formula or rule – you should always do a series of dots, connected by a line if appropriate.

If you are asked to plot a pattern, complete as much of the pattern as you are asked, and then plot the points.

The *x* axis will be number of the term, and the *y* axis the value calculated.

Patterns usually are plotted only on the positive section of the Cartesian grid.



Complete and plot:

term	1	2	3	4	5	6
value	5	7	9	11		

The missing numbers are 13 and 15, since the rule is +2

The graph is of exact points, so it is done with dots.

In some cases you will be asked to plot real life situations. In that case the relationship will usually extend between and beyond the points you are given, which can be shown by a line.

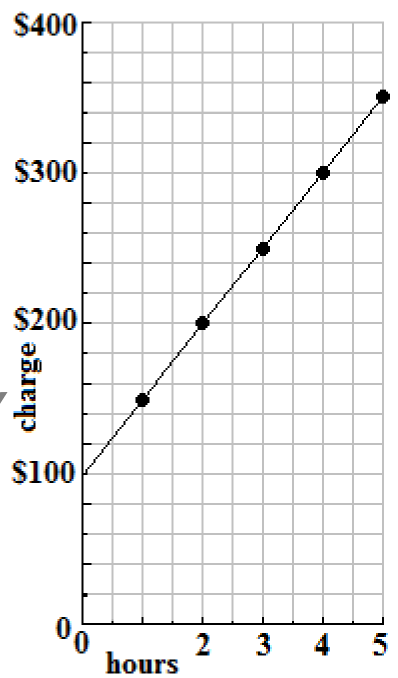
The line will be straight – do not force it go to (0, 0). If it is not straight either your scale is wrong or you have plotted points badly.

e.g. a company charges $100 plus $50 per hour.

hours	1	2	3	4	5
charge					

is filled in to become

hours	1	2	3	4	5
charge	150	200	250	300	350



The *y* axis has to be the charge, since this is being calculated from the input value of hours.

If you are given a blank grid and have to plot some pattern, relationship or formula:

- The x axis is always the input value, the y axis value is the one calculated.
- You must give a scale on each axis, but:
  - you do not need to mark every point on the scale (see the grids above).
  - the scales of the x-axis scale and y-axis can be different, if required.
- The scales must be evenly spaced all the way along.
- The axes must be labelled if not a simple x and y grid.

Technically only points should be plotted in a pattern if only whole number values are possible (e.g. the x-axis represents number of children) but a line is used to connect points if any values are possible (e.g. height).

You should select your scale so that it is easy to mark points and uses most of the space. It is not wise to use difficult spacing (try to make each grid represent a round number of values wide). Unlike Social Studies, scales should start from zero unless absolutely necessary not to.

Graphing a Line

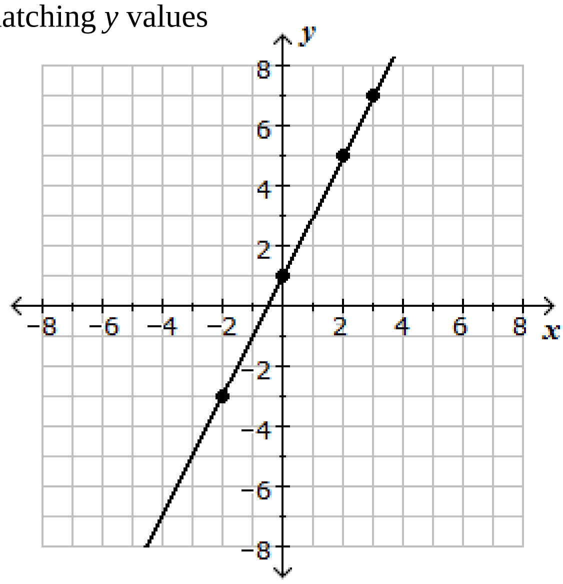
Select some starting *x* values, and put them into the formula to obtain the matching *y* values.

Plot the points obtained as (*x*, *y*), then connect them with a line.

The *x* values are chosen and put into the formula to find the matching *y* values

e.g. plotting $y = 2x + 1$

x	2*x* + 1	*y*	
0	2 × 0 + 1	1	(0, 1)
2	2 × 2 + 1	5	(2, 5)
3	2 × 3 + 1	7	(3, 7)
¯2	2 × ¯2 + 1	¯3	(¯2, ¯3)



When choosing the *x* values it pays to make them easy to calculate, especially when fractions are involved. There is also little point choosing *x* values that obviously won't fit on the graph.

e.g. plotting $y = ¾\, x + 4$

Sensible numbers

x	¼ *x* + 4	*y*
0	¼ × 0 + 4	4
4	¼ × 4 + 4	5
8	¼ × 8 + 4	6
¯4	¼ × ¯4 + 4	3

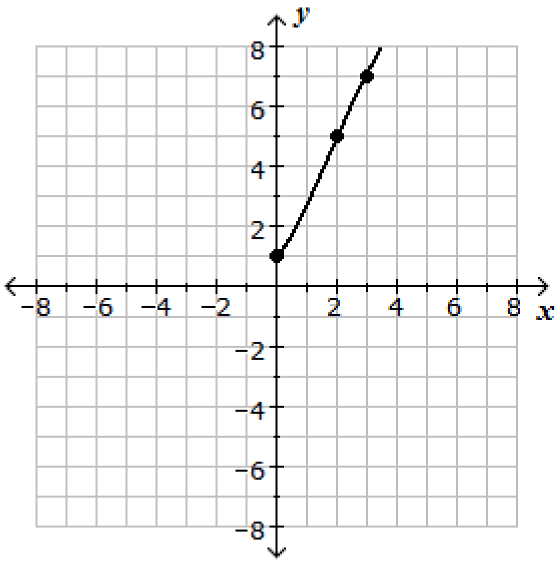
Less sensible numbers

x	¼ *x* + 4	*y*
2	¼ × 2 + 4	4.5
3	¼ × 3 + 4	4.75
100	¼ × 0 + 4	29
¯80	¼×¯80 + 4	¯16

Lines are infinite and should not be drawn as if they stop

You are marked on where your line crosses the axes (called the *x*-intercept and *y*-intercept), so you must make sure your lines cross both axes.

Although the points plotted on the graph to the right are correct, this would not be given the marks, as the line stops short.

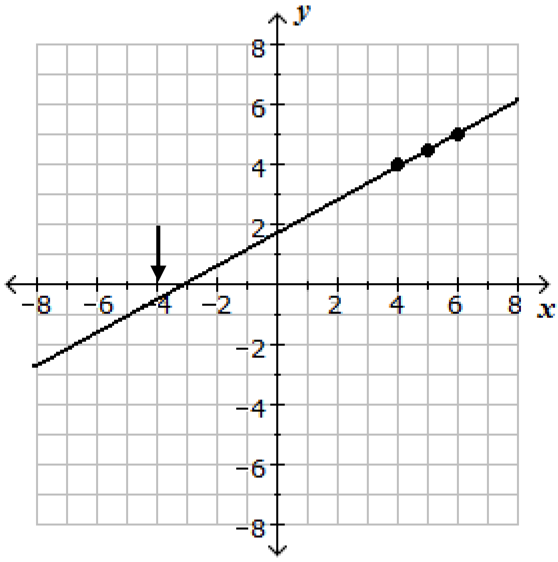


Your line should also be dead straight. Use a ruler.

If your points chosen as *x* values are too close together then it is very easy for the line to drift off over the distance, causing it to miss the correct intercepts.

e.g. plotting $y = ½\,x + 2$

x	2*x* + 3	*y*	
4	½ × 0 + 3	5	(4, 5)
5	½ × ½ + 3	5.5	(5, 5.5)
6	½ × 3 + 3	6	(6, 6)



The line is straight and looks to pass through the points. In fact it has missed the correct *y*-intercept ($^{-}4$, shown with arrow) by quite a lot. It would get no marks.

Sometimes the values chosen are off the grid given. In that case just chose more values until you have a couple which you can plot.

Some students plot a single point and then use the slope in the formula to work out the rest. This is fine, but depends totally on getting the first value correct, and can be awkward with negative and fractional values.

One recommended method is to chose start *x* values of about 4 and $^{-}4$, and then check that the line connecting them goes through the correct *y*-intercept (see below).

Finding the Formula of a Line

All angled lines can be written in the form: $y = m\,x + c$,

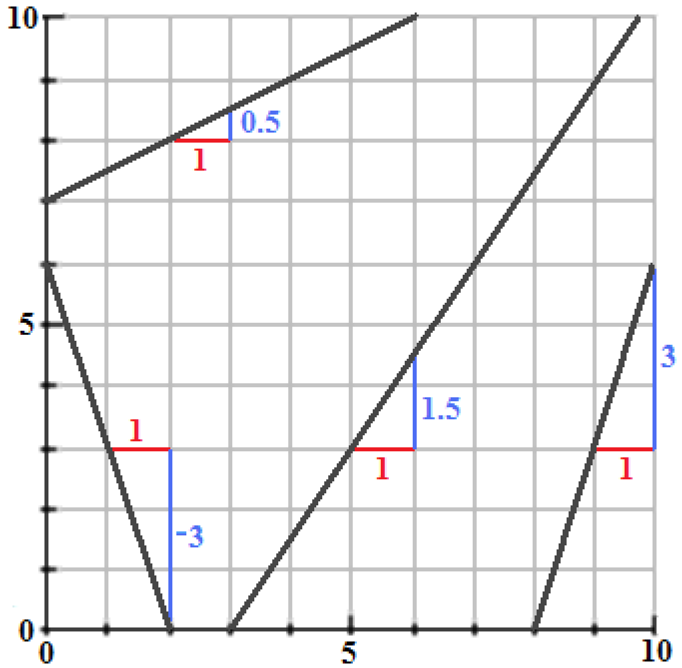
m is the <u>slope</u> or <u>gradient</u> and is equal to the amount the line rises for each one unit to the right.

c is the <u>y-intercept</u> and is equal to where the line crosses the y-axis.

The method for finding m is to select a point where the line goes through the intersection of two lines. Then calculate the amount the line goes up for one unit to the right.

If the line goes down as it moves to the right, the slope is negative.

The diagram shows the method in operation. A unit to the right is selected, and the amount the line rises for that is the slope (in blue).



Many students learn the method for finding slope as "*rise over run*". While correct, it is useless unless the student takes time to learn what "rise" and "run" mean. The *rise* is the amount a line goes up (with down being negative) for how much it goes across, the *run*, which is always to the right. The *over* means it is a division. It is better to learn it as "*slope is rise over run to the right*".

The c in the formula is where the line crosses the y-axis. Students need to be absolutely clear that it is the y-axis they use, and not the x-axis. The y-intercept is a useful check when drawing lines.

e.g. when drawing $y = 2x - 5$, the line must go through the y-axis at $^{-}5$.

Once the general method has been learnt, the usual source of errors is with negative and fractional slopes. Rigidly sticking to the method helps greatly, rather than trying to "see" the answer directly. It also helps if the student has memorised the general meaning of the slope: the larger the number the more steep it is, and rising to the right is positive.

steep uphill	gentle uphill	steep downhill	gentle downhill
= large positive	= fractional positive	=large negative	= fractional negative

Horizontal and Vertical Lines

A horizontal line has a slope of zero – for every one across the line does not move up. This means that $y = m\,x + c$ becomes a simple $y =$ y-intercept.

A vertical line has infinite slope, and so we cannot find an m for $y = m\,x + c$. Instead we use the form $x = x$-intercept.

When asked to draw a line of this form $y = c$, put the y value you have into some points with randomly selected x values.

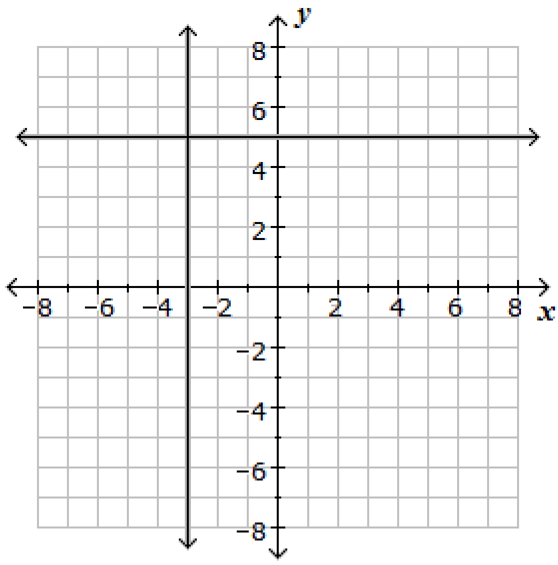
e.g. to draw $y = 6$, we select some x values – let us say, 1, 4 and ¯8. That generates the points (1, 6), (4, 6) and (¯8, 6). We connect those points to get our line.

When asked to draw a line of this form $x = c$, it is the same process but with randomly selected y values.

e.g. to draw $x = 2$, if we select y values of 2, 5 and ¯3 then we get the points (2, 2), (2, 5) and (2, ¯3). We plot those points and connect them with a line.

When finding the formula for a given line, pick some points off the line at random and see which value is staying constant.

e.g. for the graph to the right.



The horizontal line has the points (2, 5), (4, 5) and (8, 5) on it.

Looking at those we see that the y values are all 5. So our equation is $y = 5$.

The vertical line has the points (¯3, 0), (¯3, 5) and (¯3, 8) on it.

Looking at those we see that the x values are all ¯3. So our equation is $x =$ ¯3.

The biggest source of error is assuming that a line parallel to the x-axis must have the form $x = c$, when it should be $y = c$ (and vice versa for the y-axis). Sticking to a proper method helps avoid this, as does making sure to check every time you haven't made this mistake.

The x-axis is the line $y = 0$, and the y-axis is the line $x = 0$.

Other forms of a Line

Students should write the equations of lines in the slope-intercept form ($y = mx + c$) unless told otherwise, but should also recognise other forms of lines.

In general any formula with the x and y terms only at power 1 will give a line. The "standard form" for straight lines is either

$ax + by = c$ e.g. $4x + 5y = 20$

or

$ax + by + c = 0$ e.g. $4x + 5y - 20 = 0$

The reason why the "standard form" was popular was that it removes the need for any fractions, which makes printing easier.

$y = \frac{1}{2}x + 5$ becomes $x - 2y + 10 = 0$ (move y, then $\times$ 2 on both sides)

Now it is mostly kept because some relationships are most easily written that way.

If the number of *cows* plus twice the number of *bulls* add up to 100, then we can write that as a formula $c + 2b = 100$.

When asked to draw a line in these forms the long way is to rearrange the formula to the usual slope/intercept form, then put in values of x as normal. This often involves an awkward division by the b part.

$3x + 5y - 8 = 0$ becomes $5y = {}^{-}3x + 8$

becomes $y = \frac{-3}{5}x + \frac{8}{5}$

or in decimal $y = {}^{-}0.6\,x + 1.6$

The quick method is the "cover up" method. First $x = 0$ is put in, and the y intercept calculated, then $y = 0$ is put in, and the x intercept calculated. The line is drawn through these points.

$4x + 5y = 20$ becomes, when $x = 0$, $\cancel{4x +}\ 5y = 20$, so $y = 4$, and we get the point (0, 4)

$4x + 5y = 20$ becomes, when $y = 0$, $4x\ \cancel{+ 5y} = 20$, so $x = 5$, and we get the point (5, 0)

We connect (0, 4) to (5, 0) and we have the line $4x + 5y = 20$.

(It is called "cover up" because the effect of making the x then y as 0 can be achieved by covering up the relevant term with a finger, to solve more quickly.)

The gradient of a $ax + by = c$ or similar graph, is found by $m = \frac{-a}{b}$, which can be a useful check once the line is drawn. Note that if a and b are both positive, the graph has a negative gradient.

It is traditional when writing in standard form that the x term leads, and is positive (which may involve multiplying the whole equation by ${}^{-}1$). All terms should be integers.

Graphing Parabolas

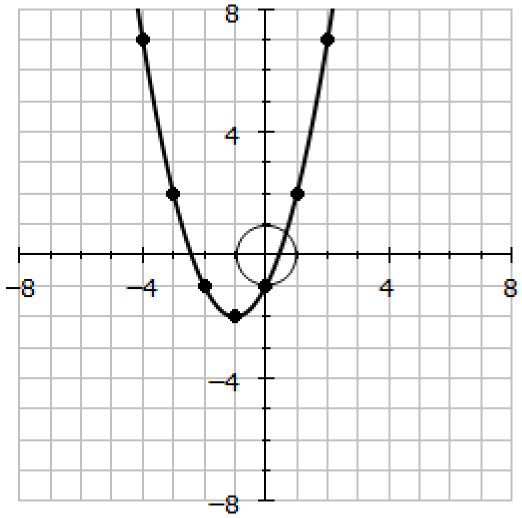
Parabolas are introduced in Year 10, and are graphs of quadratics (i.e. there is an x^2 term). They are very important for students seeking to go on to the academic strands in Years 12 and 13.

> As with lines, select x values, and put them into the formula to obtain the matching y values.
>
> Plot the points obtained as (x, y), then connect them with a curve.

The main difference from plotting lines is that until you are very confident many more x values are required. In the early days students generally have to calculate each one.

e.g. plotting $y = x^2 + 2x - 1$

x	$x^2 + 2x - 1$	y	
0	$0^2 + 2 \times 0 - 1$	$^{-}1$	$(0, {}^{-}1)$
1	$1^2 + 2 \times 1 - 1$	2	$(1, 2)$
2	$2^2 + 2 \times 2 - 1$	7	$(2, 7)$
3	$3^2 + 2 \times 3 - 1$	14	$(3, 14)$
$^{-}1$	$({}^{-}1)^2 + 2 \times {}^{-}1 - 1$	$^{-}2$	$({}^{-}1, {}^{-}2)$
$^{-}2$	$({}^{-}2)^2 + 2 \times {}^{-}2 - 1$	$^{-}1$	$({}^{-}2, {}^{-}1)$
$^{-}3$	$({}^{-}3)^2 + 2 \times {}^{-}3 - 1$	2	$({}^{-}3, 2)$
$^{-}4$	$({}^{-}4)^2 + 2 \times {}^{-}4 - 1$	7	$({}^{-}4, 7)$



The only really tricky thing with parabolas is dealing with the negative values. Of course this is easier said than done. The most common mistake is to fail to square the negative values properly. We square the whole number in any formula, including any negative sign.

If $x = {}^{-}1$ and we want x^2, then we must also square the negative $({}^{-}1)^2 = 1$. Putting -1^2 into calculators gives $^{-}1$, which leads to an incorrect graph.

The sure sign that you are graphing the negatives wrongly is when the parabola goes off track once it crosses the y-axis and gets into negative x values.

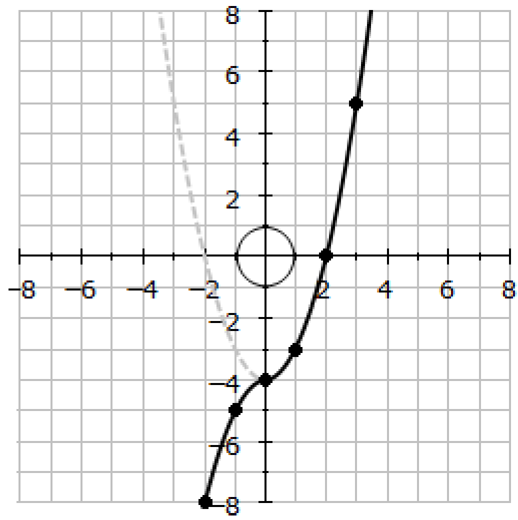
The graph to the right is $y = x^2 - 4$.

The correct path is shown dotted in grey, but the line shows someone plotting

$^{-}1^2 - 4 = -5$, rather than $({}^{-}1)^2 - 4 = {}^{-}3$,

and

$^{-}2^2 - 4 = -8$, rather than $({}^{-}2)^2 - 4 = 0$

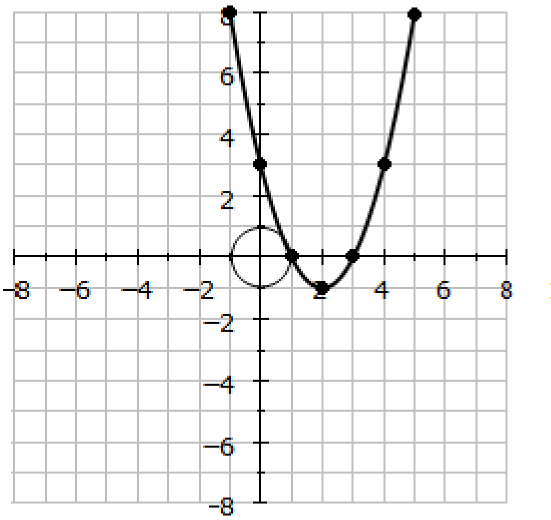


Parabolas are symmetric. Once the pattern is established on one side it should repeat on the other. In the case above the lack of symmetry into negative x-values shows immediately there is a problem.

If the formula is given in a factorised form, e.g. $y = (x - 3)(x - 1)$, nothing changes. Values of x are put into the formula, and plotted against the resulting y values. Care is needed when the results of one or both brackets are negative.

e.g. plotting $y = (x - 3)(x - 1)$

x	$(x - 3)(x - 1)$		y
0	$(0 - 3)(0 - 1)$	$(^{-}3)(^{-}1)$	3
1	$(1 - 3)(1 - 1)$	$(^{-}2)(0)$	0
2	$(2 - 3)(2 - 1)$	$(^{-}1)(1)$	$^{-}1$
3	$(3 - 3)(3 - 1)$	$(0)(2)$	0
4	$(4 - 3)(4 - 1)$	$(1)(3)$	3
5	$(5 - 3)(5 - 1)$	$(2)(4)$	8
$^{-}1$	$(^{-}1 - 3)(^{-}1 - 1)$	$(^{-}4)(^{-}2)$	8



With experience the gaps between x values become well known: from the turning point to one on either side is +1, from there +3 to go out one more, +5 etc.

If the overall x^2 term in the formula is negative, e.g. $y = 3x - x^2$, then the graph will be upside down. Extra special care is needed with negatives in this case.

Formula of a Parabolas

This is extension at Year 10. While they look fearsome, the formulas are a lot easier if students understand how they work – assuming they can follow the algebra and graphing theory required.

> If the parabola has x-intercepts of a and b, then its formula is $y = (x - a)(x - b)$.
>
> The more general form is that if the turning point is at (a, b) the formula is $y = \pm(x - a)^2 + b$

The intercept form works because x-intercepts are where $y = 0$, so are solutions to $(x - a)(x - b) = 0$.

The solid parabola goes through (2, 0) and (4, 0), so its formula is:

$$y = (x - 2)(x - 4)$$

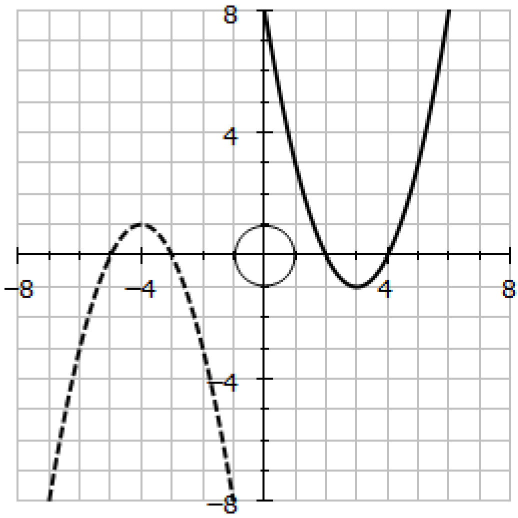
The dotted parabola goes through $(^-3, 0)$ and $(^-5, 0)$, so its formula is:

$$y = -(x - {}^-3)(x - {}^-5)$$

which becomes

$$y = -(x + 3)(x + 5)$$

(notice the leading negative, as the overall shape is upside down)



Not every parabola has intercepts (and sometimes they have them, but their exact value is unknown) so the general form is required. This works by assuming the parabola is just touching the x-axis, which would give it form $y = (x - a)^2$, and then raise or lower is by b.

The solid parabola turns at (3, 2), which gives it a formula of:

$$y = (x - 3)^2 + 2$$

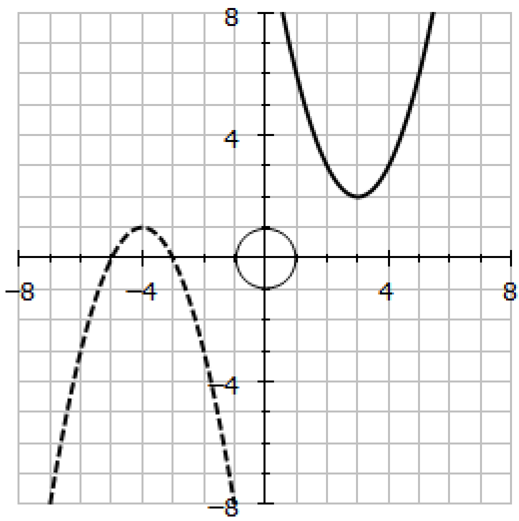
The dotted parabola turns at $(^-4, 1)$, which gives it a formula of:

$$y = -(x - {}^-4)^2 + 1$$

which becomes

$$y = -(x + 4)^2 + 1$$

(again the leading negative is needed, as the overall shape is upside down)



Students should not attempt to write equations for parabolas until they have totally mastered straight lines, or they risk confusing themselves.