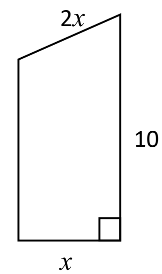
Year 12 Algebra Excellence #4

1. Write $\log_2 x + \log_8 x$ as a single log term.

2. $2x^2 - 20x + k$ has one root two-thirds the size of the other. Find k.

3. A trapezium with a rectangular end and an area of 24 cm^2 has dimensions as shown to the right, with the long non-parallel end twice the length of the right angle one.

   Calculate x.



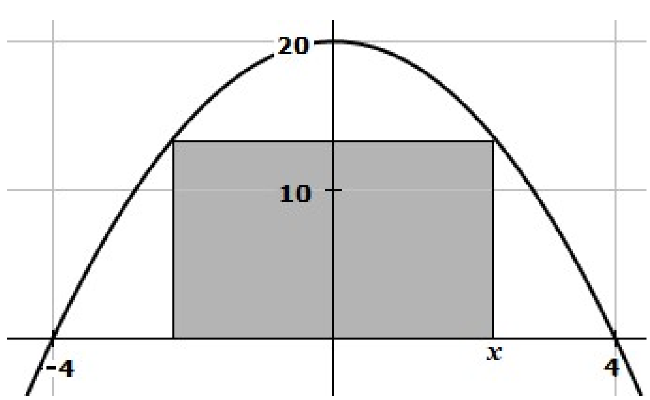
4. Find the fraction which becomes $\frac{1}{2}$ when the denominator is increased by 5 and is becomes $\frac{1}{3}$ when the numerator is decreased by 4.

5. Factorise fully: $a^2 + ab + ac + bc$.

6. Solve $5^{2x} - 5^{x+1} + 4 = 0$

7. Where does the graph of $\log_4 y = x$ cross the graph of $\log_2 y = 3x + 1$?

8. Show the shaded area, A, under this parabola is given by $A = 40x - 2.5x^3$



Answers: Year 12 Algebra Excellence #4

1. Write $\log_2 x + \log_8 x$ as a single log term.

   If we let $\log_8 x = k$ then we know that $8^k = x$

   $\Rightarrow \quad 8^k = (2^3)^k = 2^{3k} = x$ and rearranging that gives $\log_2 x = 3k = 3\log_8 x$

   $\Rightarrow \quad \log_2 x + \log_8 x = \log_2 x + \frac{1}{3}\log_2 x = \frac{4}{3}\log_2 x = \log_2 x^{\frac{4}{3}}$

   or $\log_2 x + \log_8 x = 3\log_8 x + \log_8 x = 4\log_8 x = \log_8 x^4$

   Answer: $4\log_8 x$ or $\log_8 x^4$ or $\frac{4}{3}\log_2 x$ or $\log_2 x^{\frac{4}{3}}$ (any acceptable)

2. $2x^2 - 20x + k$ has one root two-thirds the size of the other. Find k.

   $2x^2 - 20x + k = 2(x - 2r)(x - 3r)$

   (As we don't need to find r it is easier to do it this way to avoid fractions)

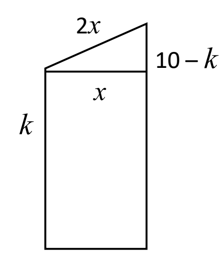
   $2x^2 - 20x + k = 2x^2 - 10r\,x + 12r^2$

   Matching coefficients: $-20x = -10r$, so r = 2. $k = 12r^2 = 12 \times 2^2$

   Answer: $k = 48$

   You can do it via: $2 \times \frac{--20+\sqrt{20^2-4\times2\times k}}{2\times2} = 3 \times \frac{--20-\sqrt{20^2-4\times2\times k}}{2\times2}$ but it's much harder

3. A trapezium with a rectangular end and an area of 24 cm^2 has dimensions as shown to the right. Calculate x.



   By Pythagoras $(10 - k)^2 = (2x)^2 - x^2$ so $10 - k = \sqrt{3}\,x$

   Area = 24 = ½ $(10 + k) \times x$ so $48 = x(10 + 10 - \sqrt{3}\,x)$

   $\sqrt{3}\,x^2 - 20x + 48 = 0 \Rightarrow x = 3.407$ or 8.144 (clearly wrong)

   Answer: $x = 3.407$

4. Find the fraction which becomes $\frac{1}{2}$ when the denominator is increased by 5 and is becomes $\frac{1}{3}$ when the numerator is decreased by 4.

   If $\frac{x}{y}$ is our answer, we are told $\frac{x}{y+5} = \frac{1}{2}$ and $\frac{x-4}{y} = \frac{1}{3}$

   Rearranging the first gives $2x = y + 5$ (multiply by both denominators)

   Similarly for the second, we get $3(x - 4) = y$

   Substituting $y = 3(x + 4)$ into the first equation gives: $2x = 3(x - 4) + 5$

   Solving gives $x = 7$. Putting that back into our first equations, we get $y = 9$.

   Answer = $\frac{7}{9}$

5. Factorise fully: $a^2 + ab + ac + bc$

$a^2 + ab + ca + cb = a(a + b) + c(a + b) = (a + c)(a + b)$.

Answer: $\mathbf{(a + c)(a + b)}$ or $(a + b)(a + c)$.

6. Solve $5^{2x} - 5^{x+1} + 4 = 0$

$\Rightarrow \quad (5^x)^2 - 5 \times 5^x + 4 = 0 \qquad \Rightarrow$ of form: $x^2 - 5x + 4 = 0$

$\Rightarrow \quad (5^x - 4)(5^x - 1) = 0$

$\Rightarrow \quad 5^x - 4 = 0$ or $5^x - 1 = 0$

$\Rightarrow \quad 5^x = 4$, so $x = \log 4 \div \log 5$ or $5^x = 1$, so $x = 0$

Answer: $\mathbf{x = 0}$ or 0.86135

7. Where does the graph of $\log_4 y = x$ cross the graph of $\log_2 y = 3x + 1$?

$\log_4 y = x$ is the same as $y = 4^x$ which is $y = 2^{2x}$

$\log_2 y = 3x + 1$ is the same as $y = 2^{3x + 1}$

So the lines cross when $y = y$, so when $2^{2x} = 2^{3x + 1}$

$2x = 3x + 1$, so $x = {}^-1$. Put this back into $y = 4^x$ gives $y = 4^{-1} = \frac{1}{4}$

Answer: lines cross at $({}^-1, 0.25)$

8. Show the shaded area, A, under this parabola is given by $A = 40x - 2.5x^3$

Formula for parabola is $y = 20 - k\,x^2$ or	$y = k\,(x + 4)(x - 4)$
Passes through (4, 0),	(0, 20) on graph
so $0 = 20 - k \times 42$	$20 = k\,(0 + 4)(0 - 4)$
$k = 1.25$	$y = {}^-1.25(x + 4)(x - 4)$
	which is $y = {}^-1.25x^2 + 20$

$A = \text{base} \times \text{height} = 2xy$

As $y = 20 - 1.25x^2$ substituting this in gives $A = 2x(20 - 1.25x^2)$

$\Rightarrow \quad A = 40x - 2.5x^3$