Year 12 Algebra Excellence #5

1. Find p if $5x^2 + px + 8$ has two equal roots.

2. $kx^2 + 3k + 22 = (4k - 14)x$ is a perfect square. Find k.

3. Solve: $9\sqrt{k} = k^2$

4. Solve: $20 \times 5^t = 8 \times 2^t$

5. Show that $\dfrac{x - 2p + 0.5}{x - q} = 2x + 1$ has more than one solution only when $q > 2\sqrt{p} - 1$

6. Find x and y which are one apart such that the difference of their reciprocals is the smaller divided by the larger.

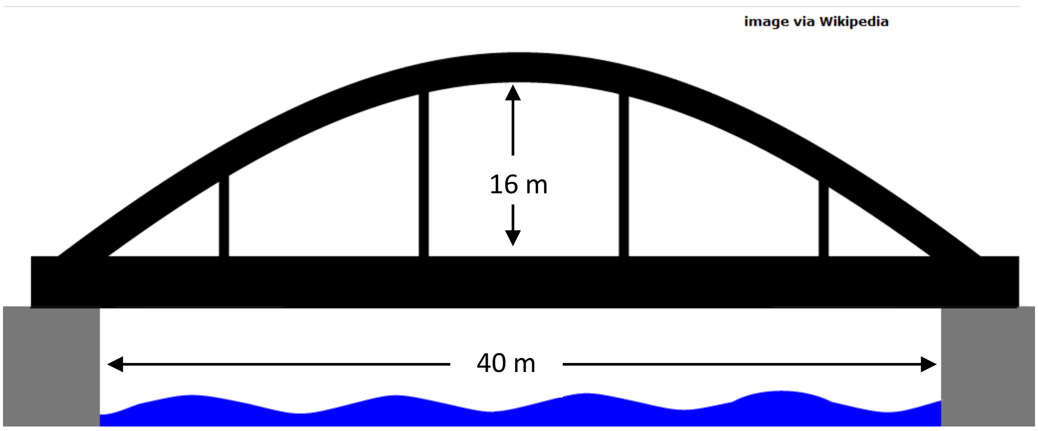
   (NB: the reciprocal of x is $\dfrac{1}{x}$)

7. Solve: $(x + 4)^2 > 7x$

8. A bridge has an inverted parabolic curve, as shown.

   The span of the parabola is 40 metres. The curve reaches 16 metres at maximum height.

   If the four struts shown are equally spaced apart (including to ends) how high are the middle struts?



Answers: Year 12 Algebra Excellence #5

1. Find p if $5x^2 + px + 8$ has two equal roots.

   $5(x - r)^2 = 5x^2 + px + 8$ or $\frac{-p+\sqrt{p^2-4\times5\times8}}{2\times5} = \frac{-p-\sqrt{p^2-4\times5\times8}}{2\times5}$

   $5x^2 - [5 \times 2r]\, x + 5r^2 = 5x^2 + px + 8$ so $p^2 - 160 = 0$

   Coefficient matching: $5r^2 = 8$, so $r = \pm\sqrt{(8 \div 5)} = \pm1.265$,

   As $p = {}^{-}10r$ we get $p = {}^{-}10 \times \pm1.265$ $p = \pm\sqrt{160}$

   p = ±12.65 (which is $\pm\sqrt{160}$)

2. $kx^2 + 3k + 22 = (4k - 14)x$ is a perfect square. Find k

   $kx^2 + 3k + 22 = (4k - 14)x$ rearranges to give $kx^2 - (4k - 14)x + (3k + 22) = 0$

   A quadratic is a perfect square when there is one root (repeated), so $b^2 - 4ac = 0$

   $(4k - 14)^2 - 4 \times k \times (3k + 22) = 0$

   $16k^2 - 112k + 196 - 12k^2 - 88k = 0$ $4k^2 - 200k + 196 = 0$

   k = 1 or 49

   Alternatively, a perfect quadratic square is of the form $(\mathrm{a}x - \mathrm{b})^2 = \mathrm{a}^2 - 2\mathrm{ab} + \mathrm{b}^2$

   so we can match the leading terms, giving $\mathrm{a}^2 = k$, $\mathrm{b}^2 = 3k + 22$ and $2\mathrm{ab} = 4k - 14$

   Since $4\,\mathrm{a}^2\,\mathrm{b}^2 = (2\mathrm{ab})^2$ we can replace $4 \times k \times (3k + 22) = (4k - 14)^2$ etc

3. Solve: $9\sqrt{k} = k^2$

   $\Rightarrow$ $9\,k^{0.5} = k^{1.5} \times k^{0.5}$ or square both sides $9^2\,k = k^4$

   $\Rightarrow$ $9\,\cancel{k^{0.5}} = k^{1.5}\,\cancel{\times k^{0.5}}$ $81k = k^4$

   $\Rightarrow$ $9 = k^{1.5}$ $81 = k^3$

   $\Rightarrow$ $k = \sqrt[1.5]{9}$ $k = \sqrt[3]{81}$

   Answer k = 4.327

4. Solve: $20 \times 5^t = 8 \times 2^t$

   $\Rightarrow$ $20 / 8 = 2^t / 5^t$

   $\Rightarrow$ $2.5 = (2/5)^t$

   $\Rightarrow$ $2.5 = 0.4^t$

   $\Rightarrow$ $\log 2.5 = t \log 0.4$

   $\Rightarrow$ $t = \log 0.4 \div \log 2.5$

   Answer t = ⁻1

5. Show that $\dfrac{x - 2p + 0.5}{x - q} = 2x + 1$ has more than one solution only when $q > 2\sqrt{p} - 1$

$\Rightarrow \quad x - 2p + 0.5 = (2x + 1)(x - q) \qquad \Rightarrow \qquad 0 = 2x^2 - 2qx + x - q - x + 2p - 0.5$

$\Rightarrow \quad 2x^2 + ({}^{-}2q)x + (2p - q - 0.5) = 0$

To have more than one solution, $b^2 - 4ac > 0$

$\Rightarrow \quad ({}^{-}2q)^2 - 4 \times 2 \times (2p - q - 0.5) > 0 \qquad \Rightarrow \qquad 4q^2 - 16p + 8q + 4 > 0$

$\Rightarrow \quad q^2 + 2q + 1 > 4p \qquad \Rightarrow \qquad (q + 1)^2 > 4p \qquad \Rightarrow \qquad \boldsymbol{q > 2\sqrt{p} - 1}$

6. Find x and y which are one apart such that the difference of their reciprocals is the smaller divided by the larger.

The reciprocal of x is $\dfrac{1}{x}$. The difference of two numbers is $x - y$.

$\Rightarrow \quad \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{x}{x+1} \qquad$ note $\dfrac{1}{x} > \dfrac{1}{x+1}$ for all $x > 0$

$\Rightarrow \quad \dfrac{x+1}{x(x+1)} - \dfrac{x}{x(x+1)} = \dfrac{x^2}{x(x+1)}$

$\Rightarrow \quad x + 1 - x = x^2$ as bottom line cancel out $\qquad \Rightarrow \qquad 1 = x^2$

$\boldsymbol{x = \pm 1}$ But −1 gives a division by zero error $\boldsymbol{x = 1}$

7. Solve: $(x - 4)^2 > 7x$

$\Rightarrow \quad x^2 - 8x + 16 > 7x \qquad \Rightarrow \qquad x^2 - 15x + 16 > 0$

$\Rightarrow \quad (x - 1.1557)(x - 13.844) > 0$

$\boldsymbol{x > 13.844}$ **or** $\boldsymbol{x < 1.1557}$

8. A bridge has an inverted parabolic curve, as shown.

The span of the parabola is 40 metres. The curve reaches 16 metres at maximum height.

If the four struts shown are equally spaced apart (including to ends) how high are the middle struts?

Set the left bottom corner of the bridge to equal (0, 0). The bottom right corner is therefore (40, 0).

The equation can then be written $y = k\,x\,(x - 40)$

We know the bridge goes through the point (20, 16) as this is the maximum.

$\Rightarrow \quad 16 = k \times 20 \times (20 - 40)$

$\Rightarrow \quad k = 16 \div -400 = {}^{-}0.04$

The struts have five spaces over 40 metres, so are 8 apart. The middle ones are therefore at the 16 m and 24 m marks.

Putting these into our equation for the curve: height = $^{-}0.04 \times 16 \times (16 - 40)$

Answer = 15.36 metres high