Y12 Calculus Excellence #1

1. Find the minimum value of $x^2 + y^2$ if $x + y = 8$.

2. A water tank has a cuboid shape, with a square base and no top.

   It has to hold 13.5 m^3 of water.

   Find the dimensions of the tank which has the least surface area.

3. A wire 180 cm long is bent into a rectangle. In order to seal it the fourth side is repeated (so there are basically five sides as shown).

   What is the maximum area of the rectangle?

4. A curve has the equation y = Ax^2 + 8x + B and passes through the point (4, 6).

   What is the curve's maximum value if it has a turning point at $x = 1.6$?

5. A train passes a point at t = 0 with a constant acceleration of $^{-}0.1$ ms^{-2} (i.e. it is slowing down). After 10 seconds the train has travelled 135 metres.

   What was its initial velocity?

6. A circular ink blot starts with a radius of 2 cm. As the ink soaks into the paper the blot expands, so that after t seconds the radius (r, in cm) is given by: $r = 2 + 0.1t$

   What is the rate of the blot's increase in area at 3 seconds?

7. $f'(x) = 8x^2$ + A for a function. It has a maximum at ($^{-}6$, 5). Where is its minimum?

8. A car sets off at a steady speed of 80 km/hr. 36 minutes later another car sets off in pursuit, starting at 70 kh/hr but accelerating from that speed steadily at +4 kh/hr^2.

   How fast is the second car going when it passes the first?

Answers: Y12 Calculus Excellence #1

1. Subbing in the y value from the second equation into $f(x) = x^2 + y^2$ gives us

   $f(x) = x^2 + (8 - x)^2 = 64 - 16x + 2x^2$ which means that $f'(x) = 4x - 16$

   Max when $f'(x) = 0$, so we solve for $0 = 4x - 16$, which gives $x = 4$ and so $y = 4$.

   Put into $x^2 + y^2$

   Minimum value = 32

2. Call length of base = x and height = h.

   We know that $x \times x \times h = 13.5$. Rearranging gives $h = \frac{13.5}{x^2}$

   Surface area = $x^2 + 4xh$ (base and four sides).

   Subbing in h we get area = $x^2 + 4x\,\frac{13.5}{x^2} = x^2 + 54x^{-1}$

   This will be at minimum when the differential = 0, so when $0 = 2x - 54x^{-2}$

   Rearranging gives $2x = 54x^{-2}$ which gives $x^3 = 27$ which gives $x = 3$

   Subbing into the $x \times x \times h = 13.5$ gives a height of $13.5 \div 9 = 1.5$.

   For smallest surface area, dimensions are 3 × 3 × 1.5 metres

3. Area = b × h

   We know that 3b + 2h = 180, so we find that: h = 90 – 1.5b

   Area = b × h = b (90 – 1.5b) = 90b – 1.5b^2

   $\frac{dA}{db}$ = 90 – 3b, which is at a maximum at the turning point when 0 = 90 – 3b

   Maximum at b = 30, so h = 45. Area = b × h = 30 × 45

   Maximum area = 1,350 cm^2

4. $\frac{dy}{dx} = 2\text{A}x + 8$ $\quad \frac{dy}{dx} = 0$ at turning point, so $0 = 2 \times \text{A} \times 1.6 + 8 \Rightarrow \text{A} = {}^{-}2.5$

   $y = {}^{-}2.5x^2 + 8x + \text{B}$ passes through (4, 6) , so $6 = {}^{-}2.5 \times 4^2 + 8 \times 4 + \text{B}$

   $\Rightarrow \quad \text{B} = 14 \Rightarrow \quad y = {}^{-}2.5x^2 + 8x + 14$

   Maximum value at turning point, so at $x = 1.6$, $y = {}^{-}2.5 \times 1.6^2 + 8 \times 1.6 + 14 = 20.4$

   The maximum value of the curve is 20.4

5. $a = -0.1$ so anti-differentiating $v = {}^{-}0.1t + \text{C}$ and again $s = {}^{-}0.05t^2 + \text{C}t + \text{c}$

At $t = 0$, $s = 0$, so c = 0, and our distance is given by $s = {}^{-}0.05t^2 + \text{C}t$

Putting in 135 metres at $t = 10$, we get $135 = {}^{-}0.05 \times 10^2 + \text{C} \times 10 \quad \Rightarrow \quad \text{C} = 14$

$v = {}^{-}0.1t + \text{C}$, so at $t = 0$, $v = {}^{-}0.1 \times 0 + 14 = 14$

The initial velocity was 14 ms^{-1}

6. Area $= \pi r^2 = \pi(2 + 0.1t)^2 = 4\pi + 0.4\pi t + 0.01\pi t^2$

rate of area's change $= \frac{dA}{dt} = 0.4\pi + 0.02\pi t$

want rate at $t = 3$ which is rate $= 0.4\pi + 0.02\pi \times 3 = 0.46\pi$

Rate = 0.46π = 1.445 cm^2/s

7. $f'(x) = 8x^2 + \text{A} \quad \Rightarrow \quad f(x) = \frac{8}{3}x^3 + \text{A}x + \text{B}$

$f'(x) = 0$ at minimum $\quad \Rightarrow \quad f'(x) = 8 \times ({}^{-}6)^2 + \text{A} = 0 \quad \Rightarrow \quad \text{A} = {}^{-}288$

Passes through $({}^{-}6, 5) \quad \Rightarrow \quad 5 = \frac{8}{3} \times ({}^{-}6)^3 + {}^{-}288 \times {}^{-}6 + \text{B} \quad \Rightarrow \quad \text{B} = {}^{-}1147$

Other turning point where $f'(x) = 0$, so when $8x^2 - 288 = 0 \quad \Rightarrow \quad$ turns at $x = {}^{+}6$

$f(6) = \frac{8}{3} \times (6)^3 + {}^{-}288 \times 6 + {}^{-}1147 = {}^{-}2{,}299$

Minimum at (6, $^{-}$2 299)

8. Note need to convert 36 minutes to 0.6 = $^6/_{10}$th of an hour.

Car 1, $v = 80$, so $s = 80t + \text{C}_1$

Car 2, $a = 4$, so $v = 4t + v_0$, as $v_0 = 70$, $v = 4t + 70$, so $s = 2t^2 + 70t + \text{C}_2$

We could set $t = 0$ when the first car sets off, so both $\text{C}_1 = 0$ and $\text{C}_2 = 0$ and replace t with $(t - 0.6)$ to account for its later start but it is much easier to set $t = 0$ when the second car sets off. At that time $\text{C}_1 = 48$ ($^6/_{10}$th of an hour at 80 km/hr) and $\text{C}_2 = 0$.

They cross when they have done the same distance

$80t + 48 = 2t^2 + 70t + 0$, which solves to give $t = {}^{-}3$ or 8. Ignore negative solution.

A $t = 8$, Car 2 is travelling $v = 4 \times 8 + 70$

The car is travelling at 102 km/hr

Solving the other way gives

$80t + 0 = 2(t - 0.6)^2 + 70(t - 0.6)$

$0 = 2t^2 - 12.4t - 41.28$, which gives $t = 8.6$ and $^{-}2.4$

$v = 4(8.6 - 0.6) + 70 = 102$ km/hr