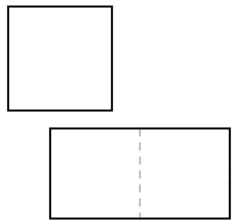
Y12 Calculus Excellence #3

1. Find the maximum value of $25y + x^2y$ if x and y add up to 10 and both are greater than zero.

2. A piece of string is 480 cm. It is cut into two pieces and a square made with one piece and a rectangle twice as wide as high made with the other.

   Find the minimum of the sum of the square's area and rectangle's area.

3. A parabola has a turning point at (4.5, 6) and a y intercept of 18. What is its equation?

4. A rocket car accelerates with a constant acceleration, the decelerates at two-thirds that acceleration (but negative) until it stops. In 10 seconds it goes 400 metres.

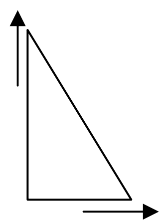
   What was its maximum speed?

5. A parachutist steps out of a balloon, and using his suit to slow the fall, has a constant acceleration of 9 ms^2.

   Two seconds later another parachutist jumps out from the same height, but goes into more of a dive to give a constant acceleration of 9.5 ms^2.

   When does the second parachutist catch up to the first?

6. A right angle triangle has a base that starts at 4 cm, and increases by 2 cm per second. The height starts at 5 cm and increases at 3 cm per second. After 10 seconds, how fast is the triangle's area increasing?

7. A parabola's tangent is parallel with the x-axis at $x = 2$ and is at 45° to the x-axis at $x = 3$. What is its tangent's gradient at $x = 8$?

8. A particle's velocity is given by $v = 0.8t - t^2$ in m s^{-1}.

   Show that in 1 second it travels a distance of 0.104 metres.

Answers: Y12 Calculus Excellence #3

1. $x + y = 10$, so $y = 10 - x$, which we sub into our value, V = $25y + x^2y$ to get

   V = $25(10 - x) + x^2(10 - x) = 250 - 25x + 10x^2 - x^3$

   V' = $0 - 25 + 20x - 3x^2$ and V' = 0 at maximum, so $3x^2 - 20x + 25 = 0$

   From calculator, $x = 5$ or $\frac{5}{3}$ and right one will be maximum as V = $^-x^3$ graph

   Substitute $x = 5$ and $y = 5$ into $25y + x^2y$

   Maximum value = 250

2. Make the square be x by x, and the rectangle be y by $2y$.

   From the length of string we know $4x + 6y = 480$, so $x = 120 - 1.5y$

   Area added, A = $x^2 + 2y^2 = (120 - 1.5y)^2 + 2y^2 = 14400 - 360y + 4.25y^2$

   A' = $^-360 + 8.5y$ and is zero at minimum, so at $y = 42.35$.

   Using $x = 120 - 1.5y$, that gives $x = 56.47$. Area = $x^2 + 2y^2 = 56.47^2 + 2 \times 42.35^2$

   Minimum area = 6776 cm^2

3. Parabola has form $y = ax^2 + bx + c$, but we are given $c = 18$ from y-intercept value.

   So $y = ax^2 + bx + 18$, and so $\frac{dy}{dx} = 2ax + b$. Turning point at (4.5. 6) where $\frac{dy}{dx} = 0$

   so $\frac{dy}{dx} = 0 = 2a \times 4.5 + b$, so $b = {}^-9a$ which we can use to substitute out b.

   $y = ax^2 - 9ax + 18$ includes point (4.5, 6) so $6 = a \times 4.5^2 - 9a \times 4.5 + 18$,

   so $a = {}^-12 \div (4.5^2 - 9 \times 4.5) = 0.5926$ and $b = {}^-9a = {}^-5.3333$

   The equation is $y = 0.5926x^2 - 5.3333x + 18$

4. Call the initial acceleration a, and it reaches top velocity of v_t

   As the deceleration is two-thirds of a, it takes 1.5 times as long to slow as speed up.

   $\Rightarrow$ it must accelerate for 4 seconds and decelerate for six.

   On the way up $v = at + c$, but $c = 0$ so $v_t = 4a$ and distance, $s = \frac{1}{2}at^2 = 8a$ after 4 sec

   On the way down, $v = v_t - \frac{2}{3}at = 4a - \frac{2}{3}at$ distance, $s = 4at - \frac{1}{3}at^2 = 12a$ after 6 sec

   Total distance = 400, so $8a + 12a = 400 \Rightarrow a = 20$, and $v_t = 4a$

   Top speed is 80 metres per second

5. #1 has a = 8, so v = 8t + C, and C = 0 as v = 0 at t = 0, so v_1 = 8t

Distance fallen, s_1 = 4t^2 + c_1

#2 has a = 9.5, so v = 9.5t + C', and C' = 0 because no initial velocity, so v_2 = 9.5t

Distance fallen, s_2 = 4.75t^2 + c_2 except that the t is this case is two seconds later.

Correction for time being two seconds later, $s_2 = 4.75(t - 2)^2$

Meet when $s_1 = s_2$ so $4t^2 = 4.75(t - 2)^2 \Rightarrow 0 = 0.75t^2 - 19t + 19$, so t = 24.29 seconds

(ignoring solution before t = 0, as meaningless)

Meet 24.29 seconds after #1 jumps

Or we can make $s_2 = 4.75t^2$ with time starting when #2 jumps, in which case #1 has

$v_1 = 8t = 8 \times 2 = 16$ at t = 0, and has travelled $s_1 = 4t^2 = 4 \times 2^2 = 16$ m already

So we can write $s_1 = 4t^2 + 16t + 16$ to compensate for first two seconds' fall

Want $s_1 = s_2$ so $4t^2 + 16t + 16 = 4.75t^2 \Rightarrow 0 = 0.75t^2 - 16t - 16$, so t = 22.29

Meet 22.29 s after #2 jumps

6. Area = ½ $b\,h$ and $b = 4 + 2t$ and $x = 5 + 3t$, so A = ½ $(4 + 2t)(5 + 3t) = 10 + 11t + 3t^2$

Rate of change of area, A' = 11 + 6t, and we want that after 10 seconds = 11 + 6 × 10

Rate = 71 cm^2/s

7. Given that $f'(2) = 0$ and $f'(3) = 1$ ($y = 1x$ is at 45° to x axis)

Let $f(x) = Ax^2 + Bx + C$, any parabola, so $f'(x) = 2Ax + B$

Since $f'(2) = 0$, $0 = 2A \times 2 + B \Rightarrow B = {}^-4A$. Since $f'(3) = 1$, $1 = 2A \times 3 + {}^-4A \Rightarrow A = 0.5$

Need gradient of tangent at x = 8, $f'(8) = 2Ax + B = 2 \times 0.5 \times 8 + {}^-4 \times 0.5 = 6$

Tangent's gradient is 6 at x = 8

8. $v(t) = 0.8t - t^2$ so displacement $s(t) = 0.4t^2 - \frac{1}{3}t^3$

Putting in t = 1 gives a displacement of $s(1) = 0.4 \times 1^2 - \frac{1}{3} \times 1^3 = 0.06666$ m

However this is the distance from the start, not the distance travelled.

At the start it has a positive velocity, until it stops at $v = 0 = 0.8t - t^2 \Rightarrow$ at t = 0.8

$s(0.8) = 0.4 \times 0.8^2 - \frac{1}{3} \times 0.8^3 = 0.085333$ m

Since it reaches 0.085333 out, and then ends up at 0.06666 it must have reversed a distance of 0.085333 – 0.066666 = 0.18666666.

So adding the 0.085333333 outwards and the 0.0186666 back

Total distance travelled is 0.104 m