Level 2 Calculus

The techniques of calculus are relatively simple. The difficulty is in recognising which technique to apply for each question.

The key is recognising when you have a function and require a gradient function, and when you have the reverse.

As part of that students need to be completely familiar with the terminology used. You can't do a question if you don't understand what it is saying.

Functions and Gradient Functions (Derivatives)

A **function** is a relationship in which each input value (x) generates only one output value (y).

If $f(x) = x^2 + 5x$ then $f(3) = 3^2 + 5 \times 3 = 24$, and the point (3, 24) is on this function.

Graphically, we can plot this relationship as $(x, f(x))$.

The gradient at any point $(x, f(x))$ is given by $f'(x)$ at that point

A **gradient function** is the relationship that gives the gradient for points on the original function.

If $f'(x) = 2x + 5$ then $f'(3) = 2 \times 3 + 5 = 11$. This means the gradient of $f(x)$ at (3, 24) is 11.

Graphically, this means the gradient at any point $(x, f(x))$ is given by $f'(x)$ at that point.

We find the gradient function by the process of differentiation, and as a result it is also called a **differential equation** or a **derivative**.

$$\textbf{If } y = f(x) \textbf{ then } \frac{dy}{dx} = f'(x)$$

If the function is written in the form $y = x^2$ then instead of $f'(x)$ we use the derivative $\frac{dy}{dx} = 2x$.

This is "y differentiated with respect to x" but is exactly the same thing as the gradient function.

For the purposes of Year 12 the only functions dealt with in calculus are the polynomials, although lots of other functions are used elsewhere in Year 12 – such as $\sqrt{x}$, $\log(x)$ and $\sin(x)$.

What Gradient is

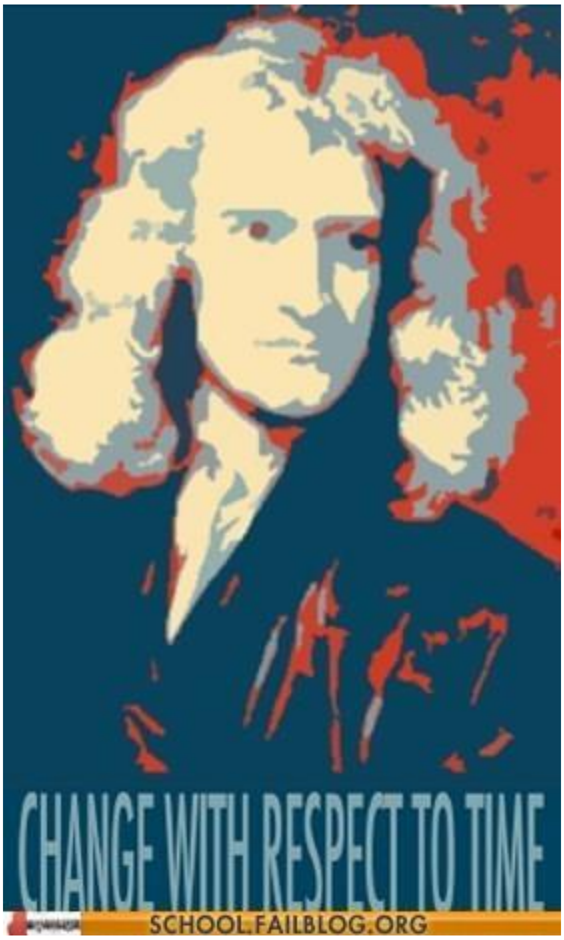
The gradient of a straight line is how fast the function goes up for every one unit to the right. That is, in other words, how fast y is changing for every unit x increases – or the rate of change of y relative to x.

We have the gradient formula for a line: $m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\Delta y}{\Delta x}$.

We can't use this for a curve because the gradient is always changing. So instead we take the **instantaneous** gradient, that is the same $\frac{\Delta y}{\Delta x}$ but when the differences, Δ, are infinitely small. In that case to indicate that the Δ is not a real amount we write the gradient as $\frac{\mathrm{d}y}{\mathrm{d}x}$, or in function notation we add a dash, $f'(x)$, each time we differentiate.

The Gradient Function $f'(x)$ and Derived Function $\frac{\mathrm{d}y}{\mathrm{d}x}$ find the following things, all identical in meaning

- The gradient at a point on a function,
- The gradient of the tangent to a point on the function,
- The rate of change of y at a point,
- The velocity at a point, if y is measure of distance and x is a measure of time.



Differentiation

The process of finding the gradient function from the function is called differentiation.

1) Each term must be separated by + or – only, so all brackets must be multiplied out. Then each term is differentiated separately.
2) If the term has a power, the differentiated term is multiplied by that number and the power then reduced by one.

   Because $x = x^1$ and $x^0 = 1$ the result of differentiating a simple x term is to remove the x.
3) A constant with no x in the term is removed. Other constants remain.

function	differentiated function	working
$f(x) = 5x^2$	$f'(x) = 10x$	$2 \times 5x^{2-1}$
$y = 0.2x^3$	$\frac{dy}{dx} = 0.6x^2$	$3 \times 0.2x^{3-1}$
$f(x) = 8x$	$f'(x) = 8$	$1 \times 8x^{1-1}$
$y = {}^{-}2x^3 + 7x$	$\frac{dy}{dx} = {}^{-}6x^2 + 7$	$3 \times {}^{-}2x^{3-1} + 1 \times 7x^{1-1}$
$f(x) = 8$	$f'(x) = 0$	constants are removed
$y = 5x^4 + 7$	$\frac{dy}{dx} = 20x^3$	$5 \times 4x^{4-1}$ + removed
$f(x) = x(2x + 2)$	$f'(x) = 4x + 2$	$f(x) = x^2 + 2x \rightarrow f'(x) = 2 \times 2x^{2-1} + 2$
$f(x) = \frac{x^2 - 5x}{4}$	$f'(x) = \frac{2x - 5}{4}$	$f(x) = \frac{1}{4}x^2 - \frac{5}{4}x \rightarrow f'(x) = \frac{2}{4}x - \frac{5}{4}$

Anti-differentiation

The process of finding the function from the gradient function is called anti-differentiation, and it recovers what the original equation must have been. Done stepwise it is:

1) Brackets must be multiplied out, as with differentiation.
2) Each term is raised by one further power of x, and the whole then divided by that number.
3) An unknown "constant of integration" **must** be added to the end of the anti-differential, to take account of any constant that was lost during the initial differentiation.

differentiated function	original function	working
$f'(x) = 10x$	$f(x) = 5x^2 + C$	$\frac{10\,x^{1+1}}{2}$
$\frac{dy}{dx} = 2x^3$	$y = 0.5x^4 + C$	$\frac{2\,x^{3+1}}{4}$
$f'(x) = 8$	$f(x) = 8x + C$	$\frac{8\,x^1}{1}$
$\frac{dy}{dx} = -2x^2 + x$	$y = \frac{-2}{3}x^3 + \frac{1}{2}x^2 + C$	$\frac{-2\,x^{2+1}}{3} + \frac{1x^{1+1}}{2}$
$f'(x) = x(2x + 3)$	$f(x) = \frac{2}{3}x^3 + 1.5\,x + C$	$f'(x) = 2x^2 + 3x \rightarrow f'(x) = \frac{2x^{2+1}}{3} + \frac{3x^{1+1}}{2}$
$y = 5x^4 - 7$	$\frac{dy}{dx} = x^5 - 7x + C$	$\frac{5\,x^{4+1}}{5} + \frac{-7x^1}{1}$
$f'(x) = \frac{x^2 - 5x}{4}$	$f(x) = \frac{2x^3 - 15x^2}{24} + C$	$\frac{x^{2+1}}{3 \times 4} + \frac{5x^{1+1}}{2 \times 4}$

Finding the Gradient at a Point

The gradient at any point $(x, f(x))$ is given by $f'(x)$ at that point.

To find the gradient at a point:
1) We differentiate our function $f(x)$ to get its gradient function $f'(x)$.
2) If not given it directly, we find the x value of the point we need.
3) We put our x value into $f'(x)$ to get the value of the gradient at that point.

e.g. Find the gradient of $f(x) = 2x^2 – 5x + 3$ when $x = 4$.

1) $f(x) = 2x^2 – 5x + 3$, so differentiating gives $f'(x) = 4x – 5$.
2) we want the gradient at $x = 4$.
3) $f'(4) = 4 \times 4 – 5 = 11$.

The gradient at $x = 4$ is 11.

The following questions are identical in working to the one above, but are expressed in slightly different ways.

- Find the gradient of $y = 2x^2 – 5x + 3$ at $x = 4$.

  Because it is $y = 2x^2 – 5x + 3$, we use $\frac{dy}{dx} = 4x – 5$, but the working is the same.
- Find the gradient of $y = 2x^2 – 5x + 3$ at (4, 15).

  The y value is irrelevant. We use the $x = 4$ value in $\frac{dy}{dx} = 4x – 5$ as before.
- Find the gradient of the tangent at (4, 15) of $f(x) = 2x^2 – 5x + 3$.

  A tangent to a graph has the gradient of the graph at that point, found using $f'(4)$.
- Find the rate of change of y at (4, 15) for $y = 2x^2 – 5x$.

  The rate of change of a function is its derived function, here $\frac{dy}{dx} = 4x – 5$.

Whenever we want to find the gradient, tangent or a rate from the equation of the amount we need to put the x value(s) into the gradient function, not just find the gradient function itself.

Sometimes we have to use knowledge of algebra or graphing to find our initial x values.

e.g. Find the rate of change of $f(x) = 2x^2 – 5x + 3$ when the function has values of 15.

$f(x) = 2x^2 – 5x + 3$, so $f(x) = 4x – 5$. But we don't have the x-values we need yet.

Solving $15 = 2x^2 – 5x + 3$ finds our x values. Those are 4 and $^{-}1.5$.

We then put them into the gradient function $f'(4) = 11$ and $f'(–1.5) = {^{-}11}$.

The rate of change when $f(x) = 15$ is 11 and $^{-}11$.

Merit: If we are asked to find the equation of a tangent at a point we need to use $y – y_1 = m(x – x_1)$.

e.g. Find the equation of the tangent of $y = x^2 – 2x + 4$ at (3, 7)

$\frac{dy}{dx} = 2x – 2$, so the gradient at $x = 3$ is $2 \times 3 – 2 = 4$.

So $y – 7 = 4(x – 3)$ which gives the tangent's equation is $y = 4x – 5$

Finding the Point for a Given Gradient

The gradient at any point $(x, f(x))$ is given by $f'(x)$ at that point.

To find the point with a given gradient:

1) We differentiate our function $f(x)$ to get its gradient function $f'(x)$.
2) We solve the equation $f'(x)$ = given gradient value to get the value (or values) for x.
3) We put our x value into $f(x)$ to get the y-value of our point.

e.g. Find the where the function $f(x) = 0.5x^2 - 2x - 5$ has a gradient of 8.

1) $f(x) = 0.5x^2 - 2x - 5$, so differentiating gives $f'(x) = x - 2$.
2) gradient is 8, so $f'(x) = 8$, so $x - 2 = 8$. Solving gives $x = 10$
3) $f(10) = 0.5 \times 10^2 - 2 \times 10$.

The gradient is 8 at (10, 30).

The following questions are identical in working to the one above, but are expressed in slightly different ways.

- Find when $y = 0.5x^2 - 2x - 5$ has a gradient of 8.

  Because $y = 0.5x^2 - 2x - 5$, we use $\frac{dy}{dx} = x - 2 = 8$, but the working is the same.

- Find when $f(x) = 2x^2 - 5x + 3$ has a tangent of form $y = 8x + c$.

  The tangent has a gradient of 8, so we need when the function has that gradient.

- Find when the rate of change of y for $y = 2x^2 - 5x$ is equal to 8.

  The rate of change of a function is its gradient, so we solve $\frac{dy}{dx} = x - 2 = 8$.

If the initial function is a cubic, then there may be two solutions for any gradient value.

e.g. Find the where the function $f(x) = x^3 - 3x - 8$ has a gradient of 24.

1) $f(x) = x^3 - 3x - 8$, so $f'(x) = 3x^2 - 3$.
2) gradient $f'(x)$ needs to be 24, so we get $24 = 3x^2 - 3$.
   Solving the quadratic (e.g. on calculator) gives $x = 3$ or $^{-}3$.
3) $f(3) = 3^3 - 3 \times 3 - 8 = 10$ and $f(3) = (^{-}3)^3 - 3 \times {}^{-}3 - 8 = {}^{-}26$.

The gradient is 24 at (3, 10) and ($^{-}$3, $^{-}$26)

Merit: If the question is posed in the form of a range of values, and particularly finding when the gradient is negative or positive, then the same procedure is done but as inequations.

e.g. Find the where the function $f(x) = 0.5x^2 - 2x - 5$ has a positive gradient.

1) $f(x) = 0.5x^2 - 2x - 5$, so $f'(x) = x - 2$.
2) gradient is positive when $f'(x) > 0$, so $x - 2 > 0$. Solving gives $x > 2$.

The gradient is positive for all values of $x > 2$.

Finding a Function from its Gradient Function

The gradient at any point $(x, f(x))$ is given by $f'(x)$ at that point.

To find the function from a given gradient function:

1) We **anti**-differentiate the gradient function $f'(x)$ to get the original function $f(x)$.
2) We calculate the value of the constant of integration by using a known point of $f(x)$.
3) We write out the original function with the value of the constant of integration.

e.g. Find the original function $f(x)$ if $f'(x) = 4x - 5$ and $f(3) = 10$.

1) $f'(x) = 4x + 5$, so anti-differentiating gives $f(x) = 2x^2 - 5x + C$.
2) $f(3) = 10$ so $2 \times 3^2 - 5 \times 3 + C = 10$, which solves for $C = 7$

The original function is $f(x) = 2x^2 - 5x + 7$.

The following questions are identical in working to the one above, but are expressed in slightly different ways.

- Find the original function $f(x)$ if $f'(x) = 4x - 5$ and that function goes through (3, 10).

  Going through (3, 10) is exactly the same as saying $f(3) = 10$.
- Find the function whose gradient at any point is equal to $4x - 5$ which goes through (3, 10).

  The gradient function is given in words, but is still the gradient function.
- Find the equation of the graph with derivative $\frac{dy}{dx} = 4x - 5$ which goes through (3, 10).

  The answer should be $y = 2x^2 - 5x + 7$, rather than $f(x) =$, but otherwise the same.
- Find the equation for y if its rate of change is $4x - 5$ and at $x = 3$ the value of $y = 10$.

  The rate of change given is equivalent to a gradient function for the amount.

It is important to not to forget to answer the question asked, which is normally to write out the equation of the original function. That means not just calculating C, but putting it back into the equation found by anti-differentiating.

Integration

Integration is technically the process of finding the area under a curve, but is also used as another term for anti-differentiation. This is why the constant found when anti-differentiating is usually known as the "constant of integration".

If C has not be calculated, we call the resulting equation the "indefinite integral".

Integration uses a long curved s (for sum) and is written: $\int 6x^2 \,.dx = 2x^3 + C$

Calculating areas under graphs used to be in Year 12, but has been recently taken out. Ignore any questions in old papers or text books that include it.

Finding a Turning Point

The turning points of a function are when $f'(x) = 0$.

To find the turning points of a function:

1) We differentiate our function $f(x)$ to get its gradient function $f'(x)$.
2) We solve the equation $f'(x) = 0$ to get the value (or values) for x.
3) We figure out if the value gives a maximum or a minimum.
4) We solve the problem, in terms of what was asked.

e.g. Find the turning points of $f(x) = x^3 + 3x^2 - 24x - 2$.

1) $f(x) = x^3 + 3x^2 - 24x - 2$, so $f'(x) = 3x^2 + 6x - 24$.
2) setting $f'(x) = 0$ we get $0 = 3x^2 + 6x - 24$. Solving gives $x = {}^{-}4$ or 2.
3) $f({}^{-}4) = ({}^{-}4)^3 + 3 \times ({}^{-}4)^2 - 24 \times ({}^{-}4) - 2 = 10$ and $f(2) = {}^{-}30$.

The turning points are at $({}^{-}4, 78)$ and $(2, {}^{-}30)$

If the initial function is a cubic, then you will find two turning points. We can tell which is the local maximum by many different methods, including:

- The y-value will be greater for the maximum.
- If the coefficient for the x^3 value is positive, then the maximum will be the left turning point, and vice versa if the x^3 value is negative.
- The value of the second differential, $f''(x)$, is negative.

The turning points give the points where:

- The graph has a local maximum or local minimum (or point of inflexion – not in Y12).
- The gradient changes from positive to negative (maximum) or *vice versa* (minimum).
- The rate of change of y with respect to x is zero.
- The amount being measured changes from increasing to decreasing (max) or *vice versa*.
- The velocity is zero, if y is measure of distance and x is a measure of time.
- The object changes direction if y is measure of distance and x is a measure of time.

Merit: The following style of questions require finding a turning point or points, but will require the answer to be interpreted in context.

- Calculate the highest point on the graph of $y = 5x - x^2$.

  The answer is the y value of the turning point.
- Find the minimum value of $f(x) = x^3 - 5x + 8$ for positive values of x.

  The answer is the y value of the turning point.
- Find where the gradient of $y = x^2 + 6x + 8$ is negative.

  The answer is a range of x values.

Whenever you are asked to find a maximum or minimum value (sometimes called "**optimisation**") you should always be looking at differentiating and then finding the turning point(s). The maximum or minimum will be the y value at those turning point.

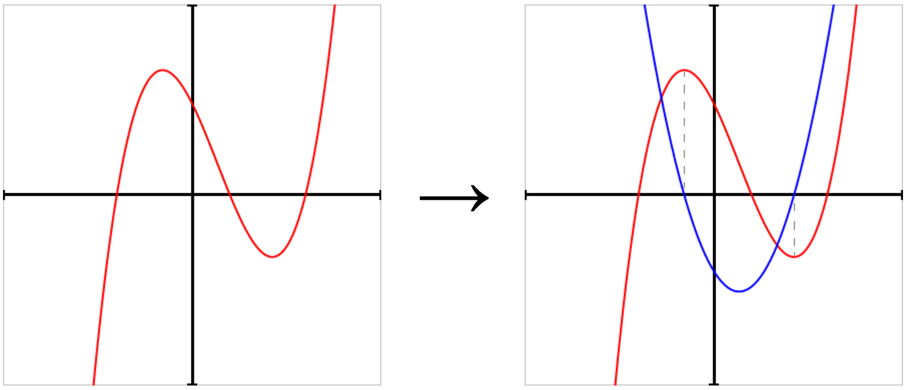
Sketching a Gradient Function

The turning points of a function are when $f'(x) = 0$.

To sketch a gradient function:

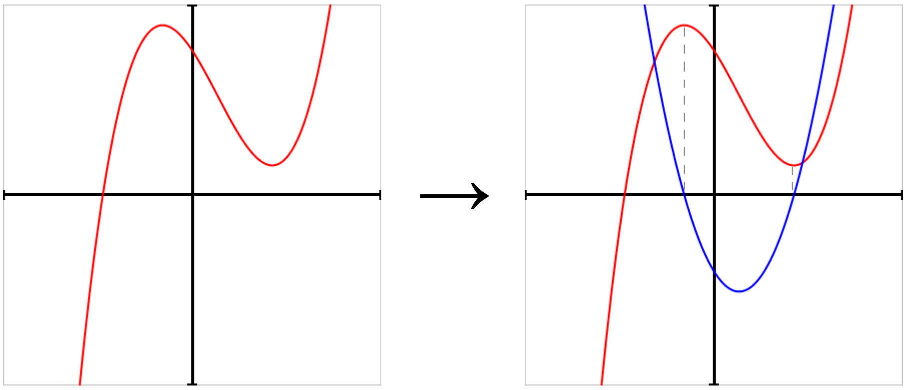
1) Mark the turning points of the original function as the gradient function's x-intercepts.
2) Differentiate the general form of the function to get the form of the gradient function.
3) Sketch in the general form so that it intercepts at the correct places and is symmetric.

e.g. Sketch the gradient function of this graph:

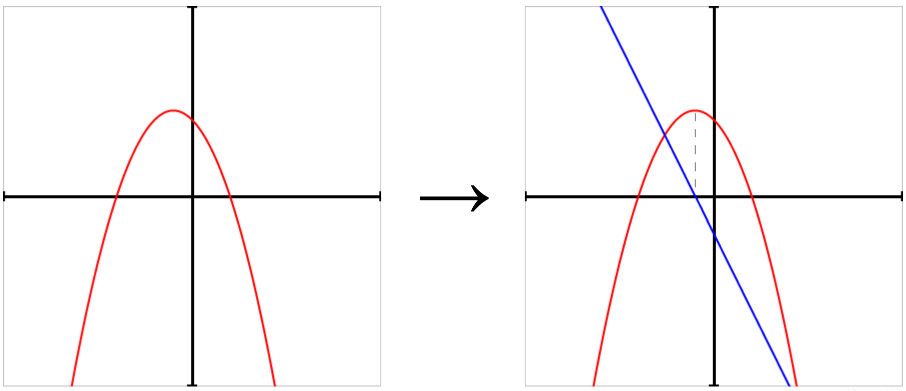


The form is $y = x^3$ so the gradient equation is of the general form $y = x^2$.
The parabola's intercepts are at the x-values of the cubic's turning points.

The x-intercepts and y-intercepts of the original graph are not relevant.



An original parabola will yield a straight line gradient function. In this case $\frac{dy}{dx}(-x^2) = -2x$



Merit: Using Variables other than *x* and *y*

The principles of calculus remain the same when the letters representing the variables or the function itself are changed. The same basic techniques apply:

- Differentiation and anti-differentiation are the same, just the *x* is replaced by another letter.
- Rates of change are given by the derivative of an amount with respect to the other variable.
- Amounts are given by anti-differentiating rates (you can spot a rate as it will be given in units with respect to time e.g. metres per second).
- A maximum or minimum will be when the derivative is = 0 (i.e. a turning point).

Some students find it helpful to relate the situation back to the simple cases. "What would I do if the variables were *x* and *y*" can be a helpful way to approach a question.

A relationship in function form can be differentiated (or anti-differentiated) in the same way, using the same basic notation and the *x* replacement behaving the same way.

e.g. Find the lowest value of $g(a) = a^3 + 3a^2 - 24a - 2$ for $a > 0$

1) $g(a) = a^3 + 3a^2 - 24a - 2$, so $g'(a) = 3a^2 + 6a - 24$.
2) setting $g'(a) = 0$ we get $0 = 3a^2 + 6a - 24$. Solving gives $a = {}^{-}4$ or 2.
3) ignore value < 0, $g(2) = 2^3 + 3 \times 2^2 - 24 \times 2 - 2 = {}^{-}30$.

The lowest value when $a > 0$ for $g(a)$ is ${}^{-}30$.

If the function is given in equation form then the $\frac{dy}{dx}$ has the variables replaced by the new ones.

e.g. Find *k* when the rate of change of *h* is 4 if we have that $h = 4k^2 - 6k$.

1) $h = 4k^2 - 6k$, so differentiating (to find rate) gives $\frac{dh}{dk}$ is $= 8k - 6$.
2) asked to find when $\frac{dh}{dk} = 4$, so $8k - 6 = 4$. Solving gives $k = 1.25$.

The rate of change of *h* is 4 when $k = 1.25$.

You can only differentiate or anti-differentiate one variable with respect to another. Excellence questions will often expect you to write your own equations, and you need to do this in such a way that the quantity you are interested in is written in terms of only one other variable.

When you are given equations where this is not true you need to rearrange to make it true. This may involve rearrangement of the original equations, often via some sort of simultaneous equation.

e.g. Find the maximum value of A if $A = xy$ and $x + 2y = 18$.

Three unknowns is an issue. The A stays, as it is what we want to maximise.

To differentiate A we need it in terms of either only *x* or only *y*.

$x + 2y = 18$ rearranges to $x = 18 - 2y$ which we can substitute into $A = xy$

$A = xy = y(18 - 2y) = 18y - 2y^2$, so differentiating A gives $\frac{dA}{dy} = 18 - 4y$.

A value is maximised when its derivative = 0, so $\frac{dA}{dy} = 0$, giving $18 - 4y = 0$.

Solving gives $y = 4.5$. Finding $x = 12 - 2x = 9$ and so $A_{max} = xy = 4.5 \times 9$.

The maximum value of A is 40.5.

Merit: Kinematics

Kinematics is the study of motion, and the questions usually look rather like ones from Physics.

Velocity is the rate of change of distance with respect to time, and acceleration is the rate of change of velocity with respect to time. Rates of change is what calculus does.

To find an equation for velocity, you differentiate the function for distance. Conversely, to find an equation for distance you anti-differentiate the equation for velocity. To find an equation for acceleration, you differentiate the function for velocity. Conversely, to find an equation for velocity you anti-differentiate that for acceleration.

Sometimes function terminology is used, except the *f* is replaced by a more appropriate letter, traditionally *a* for acceleration, *v* for velocity and *s* for distance. The variable, being time, is almost always *t*.

e.g. If velocity is given by $v(t) = 18t - t^2$, how far does the object go from $t = 0$ to $t = 6$?

Anti-differentiate *v* to get distance: $v(t) = 18t - t^2$ so $s(t) = 9t^2 - \frac{1}{3}t^3 + C$

Distance travelled is the difference between the distances at $t = 0$ and $t = 6$.

$s(6) = 9 \times 6^2 - \frac{1}{3} \times 6^3 + C = 252 + C$ and $s(0) = 9 \times 0^2 - \frac{1}{3} \times 0^3 + C = C$

$s(6) - s(0) = 252 + C - C = 252$ (note the C's always cancel for differences)

The distance travelled was 252.

If the kinematic relationship is given in equation form rather than function form the process remains the same. We will still differentiate with respect to time, usually *t*.

e.g. Distance is given by $s = 0.1t^2 + 0.5t$. How fast was the objecting going at 5 m?

Differentiate distance to get velocity: $s = 0.1t^2 + 0.5t$ so $v = \frac{ds}{dt} = 0.2t + 0.5$

Need the time. We want it when $s = 5$ so when $5 = 0.1t^2 + 0.5t$

This solves for $t = {}^{-}10$ or 5. We can ignore the negative solution here.

Putting our time into the formula for velocity: $v = 0.2 \times 5 + 0.5 = 1.5$

The object was going at a velocity of 1.5 ms^{-1} at 5m.

Sometimes kinematics involves two steps of calculus, when getting from distance to acceleration or *vice versa*.

e.g. Find the distance from the starting point after 4 seconds if acceleration is constant at $a = 10$ ms^{-2}, and the object is travelling at 5 ms^{-1} at the starting time.

Anti-differentiate *a* to get velocity: $a = 10$ so $v = 10t + C$

Told initial velocity is 5 at $t = 0$, so $v = 10t + 5$

Anti-differentiate *v* to get distance: $v = 10t + 5$ so $s = 5t^2 + 5t + C$

By definition the starting point is 0 distance at $t = 0$, so $s = 5t^2 + 5t$

Need distance at $t = 4$, so $s = 5 \times 4^2 + 5 \times 4 = 100$

The object travelled 100 m.

Note: although kinematics must have units to make sense you are not marked on them in this topic.